



Tensor products of higher almost split sequences



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ABSTRACT

We investigate how the higher almost split sequences over a tensor product of algebras are related to those over each factor. Herschend and Iyama give in [6] a criterion for when the tensor product of an n -representation finite algebra and an m -representation finite algebra is $(n + m)$ -representation finite. In this case we give a complete description of the higher almost split sequences over the tensor product by expressing every higher almost split sequence as the mapping cone of a suitable chain map and using a natural notion of tensor product for chain maps.

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1. Introduction and conventions

In the context of Auslander–Reiten theory one can study almost split sequences of modules over a finite-dimensional algebra A . These are certain short exact sequences

$$0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0$$

such that M and L are indecomposable, and it turns out that every nonprojective indecomposable module over A appears as the last term of such a sequence (and every noninjective indecomposable appears as the first term). Moreover, such sequences are determined up to isomorphism by either the first or the last term (see for reference [2]). One can do a similar construction in the context of higher dimensional Auslander–Reiten theory, at the cost of restricting to a suitable subcategory \mathcal{C} of $\text{mod } A$ that contains all injectives and all projectives. Then one gets longer so called n -almost split sequences

$$0 \rightarrow M \rightarrow X_1 \rightarrow \cdots \rightarrow X_n \rightarrow L \rightarrow 0$$

in \mathcal{C} , and again every nonprojective module in \mathcal{C} appears at the end of such a sequence and every noninjective at the start of one. Again, these sequences are determined by their first or last term (see [8,9]). One of the most basic cases where such a situation appears is when A is n -representation finite (cf. [6,8]).

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Definition. Let A be a finite-dimensional k -algebra, and let $n \in \mathbb{Z}_{>0}$. An n -cluster tilting module for A is a module $M_A \in \text{mod } A$ such that

$$\begin{aligned} \text{add } M_A &= \{X \in \text{mod } A \mid \text{Ext}_A^i(M_A, X) = 0 \text{ for every } 0 < i < n\} = \\ &= \{X \in \text{mod } A \mid \text{Ext}_A^i(X, M_A) = 0 \text{ for every } 0 < i < n\}. \end{aligned}$$

We say that A is n -representation finite if $\text{gl. dim } A \leq n$ and there exists an n -cluster tilting module for A . Then $\text{gl. dim } A = 0$ or $\text{gl. dim } A = n$.

For such algebras it is known that $\text{add } M_A$ is a subcategory of $\text{mod } A$ that admits n -almost split sequences. We call D the functor $D = \text{Hom}_k(-, k) : \text{mod } A \rightarrow A \text{ mod}$. The (higher) Auslander–Reiten translations τ_n, τ_n^- are defined as follows:

$$\begin{aligned} \tau_n &= D \text{Ext}_A^n(-, A) : \text{mod } A \rightarrow \text{mod } A \\ \tau_n^- &= \text{Ext}_A^n(DA, -) : \text{mod } A \rightarrow \text{mod } A. \end{aligned}$$

It is immediate from this definition that

$$\tau_n A = 0 = \tau_n^- DA.$$

These higher Auslander–Reiten translations behave similarly to the classical ones.

Theorem. Let A be an n -representation finite k -algebra. Let P_1, \dots, P_a be nonisomorphic representatives of the isomorphism classes of indecomposable projective right A -modules, and I_1, \dots, I_a the corresponding indecomposable injective modules. Then:

- (1) There exist positive integers l_1, \dots, l_a and a permutation $\sigma \in S_a$ (the symmetric group over a elements) such that $P_i \cong \tau_n^{l_i-1} I_{\sigma(i)}$ for every i .
- (2) There exists a unique (up to isomorphism) basic n -cluster tilting module M_A , which is given by

$$M_A = \bigoplus_{i=1}^a \bigoplus_{j=0}^{l_i-1} \tau_n^j I_{\sigma(i)}.$$

- (3) The Auslander–Reiten translations induce mutually quasi-inverse equivalences

$$\text{add}(M_A/P) \begin{matrix} \xleftarrow{\tau_n^-} \\ \xrightarrow{\tau_n} \end{matrix} \text{add}(M_A/I)$$

where $P = \bigoplus_{i=1}^a P_i$ and $I = \bigoplus_{i=1}^a I_i$.

Proof. See [9, 1.3(b)]. \square

From the last point it follows in particular that the n -cluster tilting module can be equally described by

$$M_A = \bigoplus_{i=1}^a \bigoplus_{j=0}^{l_i-1} \tau_n^{-j} P_i.$$

Definition ([6]). An n -representation finite algebra A is said to be l -homogeneous if with the above notation we have $l_1 = \dots = l_a = l$.

If A is n -representation finite, the category $\text{add } M_A$ decomposes into “slices”, in the sense that every $X \in \text{add } M_A$ can be written uniquely as $X \cong \bigoplus_{i \geq 0} X_i$, where each $X_i \in \text{add } \tau_n^{-i} A$. If A is l -homogeneous, then every slice $\text{add } \tau_n^{-j} A$, where $0 \leq j \leq l - 1$, has the same number of isomorphism classes of indecomposables.

We denote by $\mathcal{D}^b(\text{mod } A)$ the bounded derived category of $\text{mod } A$, and denote by $\varepsilon : \text{mod } A \rightarrow \mathcal{D}^b(\text{mod } A)$ the natural inclusion. The Nakayama functors

$$\nu = - \underset{A}{\overset{L}{\otimes}} DA \cong D \circ R\text{Hom}_A(-, A) : \mathcal{D}^b(\text{mod } A) \rightarrow \mathcal{D}^b(\text{mod } A)$$

$$\nu^{-1} = R\text{Hom}_{A^{op}}(D-, A) \cong R\text{Hom}_A(DA, -) : \mathcal{D}^b(\text{mod } A) \rightarrow \mathcal{D}^b(\text{mod } A)$$

are quasi-inverse equivalences that make the diagram

$$\begin{array}{ccc} \mathcal{D}^b(\text{mod } A) & \xrightarrow{\nu} & \mathcal{D}^b(\text{mod } A) \\ \uparrow & & \uparrow \\ \mathcal{K}^b(\text{proj } A) & \xrightarrow{\nu} & \mathcal{K}^b(\text{inj } A) \end{array}$$

commute (\mathcal{K}^b denotes the bounded homotopy category). If A is n -representation finite, there is a natural isomorphism of functors $\text{mod } A \rightarrow \text{mod } A$

$$\tau_n \cong H_0 \circ \nu_n \circ \varepsilon$$

where $\nu_n = \nu \circ [-n]$. For every i and for every $0 \leq j \leq l_i$, we have that $\varepsilon \tau_n^{-j} P_i = \nu_n^{-j} \varepsilon P_i$. From now on, explicit mentions of ε will be omitted for simplicity.

The definition of higher almost split sequences that is convenient to take is the following:

Definition. Let A be an n -representation finite k -algebra, and let M_A be the corresponding basic n -cluster tilting module. Let

$$0 \longrightarrow C_{n+1} \xrightarrow{f_{n+1}} C_n \longrightarrow \dots \longrightarrow C_1 \xrightarrow{f_1} C_0 \longrightarrow 0$$

be an exact sequence with terms in $\text{add } M_A$. Such a sequence is an n -almost split sequence if the following holds:

- (1) For every i , we have $f_i \in \text{rad}(C_i, C_{i-1})$.
- (2) The modules C_{n+1} and C_0 are indecomposable.
- (3) The sequence of functors from $\text{add } M_A$ to $k \text{ mod}$

$$\begin{aligned} 0 \longrightarrow \text{Hom}_A(-, C_{n+1}) &\xrightarrow{f_{n+1} \circ -} \text{Hom}_A(-, C_n) \longrightarrow \dots \\ \dots &\longrightarrow \text{Hom}_A(-, C_1) \xrightarrow{f_1 \circ -} \text{rad}_A(-, C_0) \longrightarrow 0 \end{aligned}$$

is exact (i.e. it is an exact sequence when evaluated at any $X \in \text{add } M_A$).

Theorem. Let A be an n -representation finite k -algebra, and let M_A be the corresponding basic n -cluster tilting module. Then we have the following:

(1) For every indecomposable nonprojective module $N \in \text{add } M_A$ there exists an n -almost split sequence

$$0 \longrightarrow \tau_n N \longrightarrow \cdots \longrightarrow N \longrightarrow 0,$$

and any n -almost split sequence whose last term is N is isomorphic to this one.

(2) For every indecomposable noninjective module $M \in \text{add } M_A$ there exists an n -almost split sequence

$$0 \longrightarrow M \longrightarrow \cdots \longrightarrow \tau_n^- M \longrightarrow 0,$$

and any n -almost split sequence whose first term is M is isomorphic to this one.

Proof. See [7, Theorem 3.3.1]. Notice that the term “ n -cluster tilting subcategory” has replaced “ $(n - 1)$ -orthogonal subcategory” in recent literature. \square

Remark. The usual, more general definition of n -almost split sequences that one takes requires that the condition dual to (3) holds as well (as in [9, Definition 2.1]). However, in the case we are considering (module categories over an n -representation finite algebra), the two definitions are equivalent (see [8, Proposition 2.10]).

In their paper [6], Herschend and Iyama construct a class of examples of n -representation finite algebras via tensor products, in the setting where the ground field k is perfect. Namely, they find a necessary and sufficient condition (being l -homogeneous for the same value of l) under which the tensor product $A \otimes B = A \otimes_k B$ of an n -representation finite algebra A with an m -representation finite algebra B is $(n + m)$ -representation finite. They also show that in this case every indecomposable of $\text{add } M_{A \otimes B}$ is of the form $L \otimes N$ for some indecomposables $L \in \text{add } M_A$ and $N \in \text{add } M_B$, and that $\tau_{n+m}^\pm L \otimes N \cong \tau_n^\pm L \otimes \tau_m^\pm N$. Moreover, in this case the algebra $A \otimes B$ is itself l -homogeneous.

Remark. Even though not explicitly stated in [6], necessity of the condition comes from the following observation. Let

$$M = \bigoplus_{i,j} \bigoplus_d \tau_{n+m}^{-d} P_i \otimes Q_j$$

where P_i and Q_j run over the indecomposable summands of A, B respectively. If A and B are not l -homogeneous for the same value of l , then M has either an indecomposable summand of the form $S = L \otimes J$ where J is injective and L is not, or one of the form $S = I \otimes N$ where I is injective and N is not. On the other hand, if $A \otimes B$ is $(n + m)$ -representation finite, then M is an $(n + m)$ -cluster tilting module, and hence the indecomposable injective $A \otimes B$ -modules are precisely those indecomposable direct summands $I \otimes J$ of M such that $\tau_{n+m}^- I \otimes J = 0$. Thus we reach a contradiction, since S is not injective, but $\tau_{n+m}^- S = 0$.

In this setting, if

$$0 \rightarrow L \otimes N \rightarrow \cdots \rightarrow \tau_{n+m}^- L \otimes N \rightarrow 0$$

is an $(n + m)$ -almost split sequence, then $\tau_{n+m}^- L \otimes N \cong \tau_n^- L \otimes \tau_m^- N$. On the other hand, there are n -respectively m -almost split sequences

$$0 \rightarrow L \rightarrow \cdots \rightarrow \tau_n^- L \rightarrow 0$$

and

$$0 \rightarrow N \rightarrow \cdots \rightarrow \tau_m^- N \rightarrow 0,$$

so the starting and ending points behave well with respect to tensor products. It is then a natural question to describe the relation between the sequence starting in $L \otimes N$ and the sequences starting in L and N . This is the question that we address, and we answer it in the setting where A is n -representation finite, l -homogeneous and B is m -representation finite, l -homogeneous.

For a precise statement, we need some more notation. For a preadditive category \mathcal{A} , we denote by $\mathcal{C}(\mathcal{A})$ the category of chain complexes of \mathcal{A} . If A is a k -algebra and \mathcal{A} is a full subcategory of $\text{mod } A$, we denote by $\mathcal{C}_r(\mathcal{A})$ the full subcategory of $\mathcal{C}(\mathcal{A})$ whose objects are chain complexes where the differentials are radical morphisms (i.e. $d_i \in \text{rad}(A_i, A_{i-1})$ for every i). Let \mathcal{B} be a full subcategory of $\mathcal{C}(\mathcal{A})$. We denote by $\text{Mor}(\mathcal{B})$ the category whose objects are chain maps $A_\bullet \rightarrow B_\bullet$ for $A_\bullet, B_\bullet \in \mathcal{B}$, and whose morphisms are the obvious commutative diagrams. We denote by $\text{Mor}_r(\mathcal{B})$ the full subcategory of $\text{Mor}(\mathcal{B})$ whose objects are radical chain maps $A_\bullet \rightarrow B_\bullet$ for $A_\bullet, B_\bullet \in \mathcal{B}$ (meaning that for every i the map $A_i \rightarrow B_i$ is radical). We often view finite (exact) sequences as bounded chain complexes, and unless otherwise specified the degree-0 term is the rightmost nonzero term. With this point of view in mind, we denote by \mathcal{B}^n the full subcategory of \mathcal{B} whose objects are complexes C_\bullet satisfying $C_i = 0$ for every $i < 0$ and $i > n$.

Definition. Let A be an n -representation finite k -algebra, and let $i \in \mathbb{Z}_{\geq 0}$. Let $L \in \text{add } M_A$ be indecomposable noninjective, and let C_\bullet be the corresponding n -almost split sequence. Then we say that C_\bullet starts in slice i if $L \in \text{add } \tau_n^{-i} A$.

We denote by Cone the mapping cone (see Definition 2.1). We use the symbol \otimes^T for the usual “total tensor product” bifunctor

$$- \otimes^T - : \mathcal{C}(\text{mod } A) \times \mathcal{C}(\text{mod } B) \rightarrow \mathcal{C}(\text{mod } A \otimes B)$$

induced by \otimes (see Section 3 for details). Our main result is the following:

Theorem 1.1. *Let k be a perfect field. Let A and B be n - respectively m -representation finite k -algebras. Suppose that A and B are l -homogeneous for some common l . Let $\varphi \in \text{Mor}_r(\mathcal{C}_r(\text{add } M_A))$ and let $\psi \in \text{Mor}_r(\mathcal{C}_r(\text{add } M_B))$. Suppose that $\text{Cone}(\varphi)$ and $\text{Cone}(\psi)$ are n - respectively m -almost split sequences starting in slice i for some common $i \geq 0$. Then $\text{Cone}(\varphi \otimes^T \psi)$ is an $(n + m)$ -almost split sequence.*

Remark. In Theorem 2.4 we show that every n -almost split sequence is isomorphic to $\text{Cone}(\varphi)$ for some suitable φ , so all the $(n + m)$ -almost split sequences in $\text{mod}(A \otimes B)$ are obtained by this procedure.

Remark. The sequence $\text{Cone}(\varphi \otimes^T \psi)$ starts in slice i . This is because we have (see [6])

$$\tau_n^{-i} A \otimes \tau_m^{-i} B = \tau_{n+m}^{-i} A \otimes B.$$

On the other hand, if $L \in \text{add } \tau_n^{-i} A$ and $N \in \text{add } \tau_m^{-j} B$ with $i \neq j$, then $L \otimes N \notin \text{add } M_{A \otimes B}$, so there is in principle no $(n + m)$ -almost split sequence starting in $L \otimes N$.

Remark. If we drop the condition guaranteeing that $A \otimes B$ is $(n + m)$ -representation finite, then we can perform the same construction, and we still get some sequences in $\text{mod}(A \otimes B)$ which retain some interesting properties. Similarly, one could tensor sequences that do not start in the same slice. This is a possible topic for future investigation.

In Section 2 we show that every n -almost split sequence over an n -representation finite algebra is isomorphic to the mapping cone of a suitable chain map of complexes, then we relate the property of being n -almost split to a property of the chain map. In Section 3 we define the functor \otimes^T that we have mentioned above, and we prove the main theorem. In Section 4 we compute an example where we explicitly construct a 2-almost split sequence and a 3-almost split sequence starting from a 1-representation finite algebra.

Conventions. Throughout this paper, we denote by k a perfect field (cf. [6]). All k -algebras are associative and unitary. For a ring R , we denote by $\text{mod } R$ (resp. $R \text{ mod}$) the category of finitely generated right (resp. left) R -modules. Unless otherwise specified, modules are right modules. Subcategory means full subcategory. For a k -algebra A we denote by $\text{rad}_A(-, -)$ the subfunctor of $\text{Hom}_A(-, -)$ defined by

$$\text{rad}_A(X, Y) = \{f \in \text{Hom}_A(X, Y) \mid \text{id}_X - g \circ f \text{ is invertible } \forall g \in \text{Hom}_A(Y, X)\}$$

for all A -modules X, Y (see [2, Appendix 3]). Thus $\text{rad}_A(-, -)$ is biadditive, and for two indecomposable modules $X \not\cong Y$ we have $\text{rad}_A(X, Y) = \text{Hom}_A(X, Y)$. Moreover, for an indecomposable module X we have that $\text{rad}_A(X) := \text{rad}_A(X, X)$ is the Jacobson radical of the algebra $\text{End}_A(X)$. We denote by $S_A(X, Y)$ the quotient $S_A(X, Y) = \text{Hom}_A(X, Y) / \text{rad}_A(X, Y)$ (and sometimes write only $S_A(X)$ instead of $S_A(X, X)$). To simplify the notation, we sometimes omit the reference to the algebra when this is clear from the context (writing for instance Hom instead of Hom_A). For the rest of this paper, fix finite-dimensional k -algebras A and B , where A is n -representation finite and B is m -representation finite. Set $\mathcal{A} = \text{add } M_A$, $\mathcal{B} = \text{add } M_B$, $\mathcal{A}_i = \text{add } \tau_n^{-i} A$ for $i \geq 0$, and $\mathcal{B}_j = \text{add } \tau_m^{-j} B$ for $j \geq 0$.

2. n -Almost split sequences as mapping cones

2.1. Preliminaries

If A is n -representation finite, then the morphisms in \mathcal{A} are “directed” with respect to the action of τ_n^- . More precisely, we have the following:

Proposition 2.1. *Let A be an n -representation finite k -algebra. Let $M \in \mathcal{A}_i$ and $N \in \mathcal{A}_j$ with $i > j$. Then*

$$\text{Hom}_A(M, N) = 0.$$

Proof. It is enough to check the result for M, N indecomposable, i.e. $M \cong \tau_n^{-i} P_1$ and $N \cong \tau_n^{-j} P_2$ for some indecomposable projectives $P_1, P_2 \in \text{add } A$. We have

$$\begin{aligned} \text{Hom}_A(M, N) &= \text{Hom}_{\mathcal{D}^b(\text{mod } A)}(M, N) = \text{Hom}_{\mathcal{D}^b(\text{mod } A)}(\nu_n^{-i} P_1, \nu_n^{-j} P_2) = \\ &= \text{Hom}_{\mathcal{D}^b(\text{mod } A)}(P_1, \nu_n^{i-j} P_2). \end{aligned}$$

In particular, $\text{Hom}_A(M, N)$ is a direct summand of (with the previous notation)

$$\begin{aligned} \bigoplus_{i=1}^a \text{Hom}_{\mathcal{D}^b(\text{mod } A)}(P_i, \nu_n^{i-j} P_2) &= \text{Hom}_{\mathcal{D}^b(\text{mod } A)}(A, \nu_n^{i-j} P_2) = \\ &= H_0(\nu_n^{i-j} P_2) = \tau_n^{i-j} P_2 = 0 \end{aligned}$$

since $i > j$ and P_2 is projective, so we are done. \square

Remark 2.1. For $n = 1$, this is a special case of [1, Corollary VIII.1.4], since “1-representation finite” means “hereditary and representation finite”.

We will be interested in checking whether a given complex is an n -almost split sequence, and for this purpose it is convenient to take a slightly different point of view on the definition of n -almost splitness. Namely, fix an object $X \in \mathcal{A}$. We can define a functor $F_X : \mathcal{C}_r(\mathcal{A}) \rightarrow \mathcal{C}(k \text{ mod})$ by mapping

$$C_\bullet = \dots \xrightarrow{f_{i+1}} C_i \xrightarrow{f_i} \dots \xrightarrow{f_1} C_0 \xrightarrow{f_0} \dots$$

to

$$F_X(C_\bullet) = \dots \xrightarrow{f_{i+1} \circ -} \text{Hom}(X, C_i) \xrightarrow{f_i \circ -} \dots \xrightarrow{f_1 \circ -} \text{rad}(X, C_0) \xrightarrow{f_0 \circ -} \dots$$

(that is, F_X is the subfunctor of $\text{Hom}(X, -)$ given by replacing $\text{Hom}(X, C_0)$ with $\text{rad}(X, C_0)$). This is well defined since f_1 is a radical morphism, hence the image of $f_1 \circ -$ lies in $\text{rad}(X, C_0)$. Then for a complex $C_\bullet \in \mathcal{C}_r(\mathcal{A})$ such that $C_i = 0$ for $i > n + 1$ and $i < 0$, saying that it is an n -almost split sequence is equivalent to saying that C_{n+1} and C_0 are indecomposable, C_\bullet is exact, and $F_X(C_\bullet)$ is exact for every $X \in \mathcal{A}$ (or equivalently, for every indecomposable $X \in \mathcal{A}$). Similarly, we can define a subfunctor G_X of the contravariant functor $\text{Hom}(-, X) : \mathcal{C}_r(\mathcal{A}) \rightarrow \mathcal{C}(k \text{ mod})$ by mapping C_\bullet to

$$G_X(C_\bullet) = \dots \xrightarrow{- \circ f_0} \text{Hom}(C_0, X) \xrightarrow{- \circ f_1} \dots \xrightarrow{- \circ f_{n+1}} \text{rad}(C_{n+1}, X) \xrightarrow{- \circ f_{n+2}} \dots$$

This is again well defined, and if $C_\bullet \in \mathcal{C}_r(\mathcal{A})$ is n -almost split then $G_X(C_\bullet)$ is exact for every $X \in \mathcal{A}$ (cf. [8, Proposition 2.10]).

2.2. From sequences to cones

Definition 2.1. Let \mathcal{D} be an abelian category. Let $A_\bullet \in \mathcal{C}(\mathcal{D})$ with differentials $d_i : A_i \rightarrow A_{i-1}$. For any $m \in \mathbb{Z}$, the *shift* $A[m]_\bullet$ of A_\bullet is the complex with objects $A[m]_i = A_{i+m}$ and differentials $d[m]_i : A[m]_i \rightarrow A[m]_{i-1}$ given by $d[m]_i = (-1)^m d_{i+m}$ for every i .

Let (A_\bullet, d_\bullet^A) and (B_\bullet, d_\bullet^B) be complexes in $\mathcal{C}(\mathcal{A})$. Let $f : A_\bullet \rightarrow B_\bullet$ be a morphism of complexes with components $f_i : A_i \rightarrow B_i$. The *shift* of f is the morphism $f[m] : A_\bullet[m] \rightarrow B_\bullet[m]$ with components $f[m]_i = f_{i+m}$. Thus $[m]$ is an endofunctor on $\mathcal{C}(\mathcal{D})$. The *mapping cone* $\text{Cone}(f)$ of f is the complex with objects

$$\text{Cone}(f)_i = A[-1]_i \oplus B_i$$

and differentials

$$d_i^{\text{Cone}(f)} = \begin{bmatrix} d[-1]_i^A & 0 \\ f[-1]_i & d_i^B \end{bmatrix}.$$

Lemma 2.2. Let \mathcal{D} be an abelian category, and let f be a morphism of complexes in $\mathcal{C}(\mathcal{D})$. Then $\text{Cone}(f)$ is exact if and only if f is a quasi-isomorphism.

Proof. This follows straight from [5, III.18]. \square

Let A be n -representation finite, and let

$$C_\bullet = 0 \longrightarrow C_{n+1} \xrightarrow{f_{n+1}} C_n \longrightarrow \dots \longrightarrow C_1 \xrightarrow{f_1} C_0 \longrightarrow 0$$

be an n -almost split sequence starting in slice i_0 for some $i_0 \in \mathbb{Z}_{\geq 0}$. Then we can decompose the modules appearing in the sequence according to the slice decomposition of \mathcal{A} , i.e. we write

$$C_m = \bigoplus_{j \geq 0} B_m^j$$

with $B_m^j \in \mathcal{A}_j$ for every m, j . We know that $C_{n+1} \in \mathcal{A}_{i_0}$ and $C_0 \in \mathcal{A}_{i_0+1}$ are indecomposable. A first result, which can be seen as a generalisation of [1, Lemma VIII.1.8(b)], is the following:

Lemma 2.3. *With the above notation, we have*

$$B_m^j = 0 \text{ for any } m \text{ and for } j \notin \{i_0, i_0 + 1\}.$$

Proof. To reach a contradiction, suppose that the claim is false. Then there is $B_q^j \neq 0$ with $j \notin \{i_0, i_0 + 1\}$. Suppose $j > i_0 + 1$, and pick j maximal such. We can assume q minimal for that value of j , i.e. $B_{q-p}^j = 0$ for all $p > 0$. Notice that since $C_0 = B_0^{i_0+1}$ it follows that $q > 0$. We want to prove that C_\bullet cannot be n -almost split in this case, and it is enough to show that $F_{B_q^j}(C_\bullet)$ is not exact. By Proposition 2.1,

$$\text{Hom}(B_{p'}^j, B_p^i) = 0$$

for every p, p' and for every $i < j$. By maximality of j , we get that B_\bullet^j is a subcomplex of C_\bullet , and

$$F_{B_q^j}(C_\bullet) = F_{B_q^j}(B_\bullet^j).$$

Since q is minimal and $q > 0$ we can write explicitly

$$F_{B_q^j}(C_\bullet) = \cdots \longrightarrow \text{Hom}(B_q^j, B_m^j) \longrightarrow \cdots \xrightarrow{d} \text{Hom}(B_q^j, B_q^j) \longrightarrow 0.$$

The map d in this sequence is composition with a radical morphism, so in particular it cannot be surjective on $\text{Hom}(B_q^j, B_q^j)$. The sequence is then not exact and we have proved that $B_m^j = 0$ for $j > i_0 + 1$.

Suppose now that $j < i_0$, and pick j minimal such. We can assume that q is maximal for that j , i.e. $B_{q+p}^j = 0$ for all $p > 0$. Notice that since $C_{n+1} = B_{n+1}^{i_0}$ it follows that $q < n + 1$. We prove that C_\bullet is not n -almost split in this case by showing that $G_{B_q^j}(C_\bullet)$ is not exact. Again by Proposition 2.1 we know that

$$\text{Hom}(B_p^i, B_{p'}^j) = 0$$

for all p, p' if $i > j$. Then by minimality of j and maximality of q we get

$$G_{B_q^j}(C_\bullet) = \cdots \longrightarrow \text{Hom}(B_m^j, B_q^j) \longrightarrow \cdots \xrightarrow{d'} \text{Hom}(B_q^j, B_q^j) \longrightarrow 0$$

and d' cannot be surjective, contradiction. Hence we have proved that $B_m^j = 0$ for $j < i_0$, which completes the proof. \square

Theorem 2.4. *Let A be an n -representation finite k -algebra, and let $i_0 \in \mathbb{Z}_{\geq 0}$. Let $C_{n+1} \in \mathcal{A}_{i_0}$ be indecomposable noninjective, and let*

$$C_\bullet = 0 \longrightarrow C_{n+1} \xrightarrow{f_{n+1}} C_n \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{f_1} C_0 \longrightarrow 0$$

be the corresponding n -almost split sequence. Then there are complexes $A_{\bullet}^0 \in \mathcal{C}_r(\mathcal{A}_{i_0})$, $A_{\bullet}^1 \in \mathcal{C}_r(\mathcal{A}_{i_0+1})$, and a radical morphism of complexes $\varphi : A_{\bullet}^0 \rightarrow A_{\bullet}^1$, such that $C_{\bullet} \cong \text{Cone}(\varphi)$ in $\mathcal{C}(\mathcal{A})$.

Proof. By Lemma 2.3 we can rewrite the complex C_{\bullet} as

$$C_m = B_m^{i_0} \oplus B_m^{i_0+1}$$

where $B_m^{i_0} \in \mathcal{A}_{i_0}$ and $B_m^{i_0+1} \in \mathcal{A}_{i_0+1}$ for every m . Moreover,

$$f_m = \begin{bmatrix} b_m^{i_0} & \xi_m \\ \gamma_m & b_m^{i_0+1} \end{bmatrix} : C_m \rightarrow C_{m-1}$$

has components $b_m^{i_0} : B_m^{i_0} \rightarrow B_{m-1}^{i_0}$, $\xi_m : B_m^{i_0+1} \rightarrow B_{m-1}^{i_0}$, $\gamma_m : B_m^{i_0} \rightarrow B_{m-1}^{i_0+1}$, and $b_m^{i_0+1} : B_m^{i_0+1} \rightarrow B_{m-1}^{i_0+1}$. Notice that by Proposition 2.1 it follows that $\xi_m = 0$ for all m . Define $A_m^0 = B_{m+1}^{i_0}$, $d_m^{A^0} = -b_{m+1}^{i_0}$, $A_m^1 = B_m^{i_0+1}$, $d_m^{A^1} = b_m^{i_0+1}$ and $\varphi_m = -\gamma_{m+1} : A_m^0 \rightarrow A_m^1$. Then $\varphi : A_{\bullet}^0 \rightarrow A_{\bullet}^1$ is a chain map since

$$d_m^{A^1} \varphi_m = -b_m^{i_0+1} \gamma_{m+1} = \gamma_m b_{m+1}^{i_0} = \varphi_{m-1} d_m^{A^0}$$

where the equality

$$b_m^{i_0+1} \gamma_{m+1} + \gamma_m b_{m+1}^{i_0} = 0$$

comes from the fact that C_{\bullet} is a complex. Moreover, $C_{\bullet} \cong \text{Cone}(\varphi)$ and we are done. \square

Remark 2.2. In [9, Proposition 3.23] Iyama constructed certain n -almost split sequences as mapping cones of chain maps. Our Theorem 2.4 states that in the n -representation finite case, every n -almost split sequence can in fact be realised as a mapping cone.

Given that n -almost split sequences are determined up to isomorphism by their endpoints, it is interesting to address the issue of uniqueness of the map φ . Since we are not going to need it in what follows, we do not investigate this in detail. We present however a result:

Proposition 2.5. Let A be an n -representation finite algebra. Let $A_{\bullet}^0, B_{\bullet}^0 \in \mathcal{C}(\mathcal{A}_{i_0})$, $A_{\bullet}^1, B_{\bullet}^1 \in \mathcal{C}(\mathcal{A}_{i_0+1})$. Let $\varphi : A_{\bullet}^0 \rightarrow A_{\bullet}^1$ and $\psi : B_{\bullet}^0 \rightarrow B_{\bullet}^1$ be chain maps. Then the following are equivalent:

- (1) $\text{Cone}(\varphi) \cong \text{Cone}(\psi)$ in $\mathcal{C}(\mathcal{A})$.
- (2) There are isomorphisms of complexes $f : A_{\bullet}^0 \rightarrow B_{\bullet}^0$, $g : A_{\bullet}^1 \rightarrow B_{\bullet}^1$ such that the diagram

$$\begin{array}{ccc} A_{\bullet}^0 & \xrightarrow{f} & B_{\bullet}^0 \\ \varphi \downarrow & & \downarrow \psi \\ A_{\bullet}^1 & \xrightarrow{g} & B_{\bullet}^1 \end{array}$$

commutes in the homotopy category $\mathcal{K}(\mathcal{A})$.

Proof. Let us begin by some observations. Let

$$\alpha_m = \begin{bmatrix} a_m & r_m \\ q_m & b_m \end{bmatrix} : A_{m-1}^0 \oplus A_m^1 \rightarrow B_{m-1}^0 \oplus B_m^1$$

be a morphism of modules. Notice that by Proposition 2.1, we have $r_m = 0$. Observe now that

$$\begin{aligned}
 & (\alpha_m) \text{ defines a chain map } \alpha : \text{Cone}(\varphi) \rightarrow \text{Cone}(\psi) \\
 \Leftrightarrow & \begin{bmatrix} a_{m-1} & 0 \\ q_{m-1} & b_{m-1} \end{bmatrix} \begin{bmatrix} -d_{m-1}^{A^0} & 0 \\ \varphi_{m-1} & d_m^{A^1} \end{bmatrix} = \begin{bmatrix} -d_{m-1}^{B^0} & 0 \\ \psi_{m-1} & d_m^{B^1} \end{bmatrix} \begin{bmatrix} a_m & 0 \\ q_m & b_m \end{bmatrix} \text{ for all } m \\
 \Leftrightarrow & \begin{cases} a_{m-1}d_{m-1}^{A^0} = d_{m-1}^{B^0}a_m & \text{for all } m \\ b_{m-1}d_m^{A^1} = d_m^{B^1}b_m & \text{for all } m \\ b_{m-1}\varphi_{m-1} = \psi_{m-1}a_m + d_m^{B^1}q_m + q_{m-1}d_{m-1}^{A^0} & \text{for all } m \end{cases} \\
 \Leftrightarrow & \begin{cases} (a_m) \text{ defines a chain map } a : A^0[-1]_{\bullet} \rightarrow B^0[-1]_{\bullet} \\ (b_m) \text{ defines a chain map } b : A^1_{\bullet} \rightarrow B^1_{\bullet} \\ b_{m-1}\varphi_{m-1} = \psi_{m-1}a_m + d_m^{B^1}q_m + q_{m-1}d_{m-1}^{A^0} & \text{for all } m. \end{cases}
 \end{aligned}$$

Now let us prove (1) \Rightarrow (2). Use the same notation as above, and assume that α is an isomorphism. That means that α_m is an isomorphism for every m . Since $A_{m-1}^0 \in \mathcal{A}_{i_0}$ and $B_m^1 \in \mathcal{A}_{i_0+1}$, it follows that no indecomposable direct summand of A_{m-1}^0 can be isomorphic to a direct summand of B_m^1 , hence $\text{Hom}(A_{m-1}^0, B_m^1) = \text{rad}(A_{m-1}^0, B_m^1)$. In particular we have that q_m is a radical map. Since α_m has an inverse, both a_m and b_m have inverses modulo radical morphisms. This means that there are $x : B_{m-1}^0 \rightarrow A_{m-1}^0, y : B_m^1 \rightarrow A_m^1$ such that

$$\begin{aligned}
 & a_mx - \text{id}_{B_{m-1}^0}, \\
 & xa_m - \text{id}_{A_{m-1}^0}, \\
 & b_my - \text{id}_{B_m^1}, \\
 & yb_m - \text{id}_{A_m^1}
 \end{aligned}$$

are radical morphisms. In particular a_mx, xa_m, b_my, yb_m are all invertible, hence a_m and b_m are isomorphisms. By the above observations, $a[1]$ and b are well-defined isomorphisms of complexes, and since

$$\left(d_m^{B^1}q_m + q_{m-1}d_{m-1}^{A^0} \right) : A_{\bullet}^0 \rightarrow B_{\bullet}^1$$

is null-homotopic we obtain that the diagram

$$\begin{array}{ccc}
 A_{\bullet}^0 & \xrightarrow{a[1]} & B_{\bullet}^0 \\
 \varphi \downarrow & & \downarrow \psi \\
 A_{\bullet}^1 & \xrightarrow{b} & B_{\bullet}^1
 \end{array}$$

commutes in $\mathcal{K}(\mathcal{A})$ as required.

Let us now prove (2) \Rightarrow (1). Since the diagram commutes in $\mathcal{K}(\mathcal{A})$, there is a homotopy $(q_m : A_{m-1}^0 \rightarrow B_m^1)$ such that

$$b_{m-1}\varphi_{m-1} = \psi_{m-1}a_m + d_m^{B^1}q_m + q_{m-1}d_{m-1}^{A^0} \text{ for all } m.$$

By the above observations, setting for every m

$$\alpha_m = \begin{bmatrix} f_{m-1} & 0 \\ q_m & g_m \end{bmatrix} : A_{m-1}^0 \oplus A_m^1 \rightarrow B_{m-1}^0 \oplus B_m^1$$

defines a chain map $\alpha : \text{Cone}(\varphi) \rightarrow \text{Cone}(\psi)$. It remains to check that α is an isomorphism, which amounts to checking that α_m is invertible for all m . Since we are assuming that f and g are isomorphisms, we can define for every m

$$\beta_m = \begin{bmatrix} f_{m-1}^{-1} & 0 \\ -g_m^{-1}q_m f_{m-1}^{-1} & g_m^{-1} \end{bmatrix} : B_{m-1}^0 \oplus B_m^1 \rightarrow A_{m-1}^0 \oplus A_m^1.$$

It is then a straightforward computation to check that β_m is the inverse of α_m , and we are done. \square

2.3. From cones to sequences

Since we can realise any n -almost split sequence as $\text{Cone}(\varphi)$ for some φ , it makes sense to relate the property of being n -almost split to the properties of φ . Let us introduce some more notation. For a given $X \in \mathcal{A}$, we define a functor $\tilde{F}_X : \text{Mor}_r(\mathcal{C}_r(\mathcal{A})) \rightarrow \text{Mor}(\mathcal{C}(k \text{ mod}))$ by mapping $\varphi : A_\bullet \rightarrow B_\bullet$ to

$$\tilde{F}_X(\varphi) = \varphi \circ - : \text{Hom}(X, A_\bullet) \rightarrow F_X(B_\bullet),$$

where $\text{Hom}(X, A_\bullet)$ denotes the complex $\cdots \rightarrow \text{Hom}(X, A_i) \rightarrow \text{Hom}(X, A_{i-1}) \rightarrow \cdots$. This is well defined because $\varphi_0 \in \text{rad}(A_0, B_0)$.

Consider the mapping cone functor $\text{Cone} : \text{Mor}_r(\mathcal{C}_r(\mathcal{A})) \rightarrow \mathcal{C}(\mathcal{A})$. By definition, this factors through the inclusion $\mathcal{C}_r(\mathcal{A}) \rightarrow \mathcal{C}(\mathcal{A})$, and we still denote by Cone the corresponding functor $\text{Cone} : \text{Mor}_r(\mathcal{C}_r(\mathcal{A})) \rightarrow \mathcal{C}_r(\mathcal{A})$. We also denote by Cone the mapping cone functor $\text{Cone} : \text{Mor}(\mathcal{C}(k \text{ mod})) \rightarrow \mathcal{C}(k \text{ mod})$.

Lemma 2.6. *With the above notation, we have that the diagram*

$$\begin{array}{ccc} \text{Mor}_r(\mathcal{C}_r^n(\mathcal{A})) & \xrightarrow{\tilde{F}_X} & \text{Mor}(\mathcal{C}^n(k \text{ mod})) \\ \text{Cone} \downarrow & & \downarrow \text{Cone} \\ \mathcal{C}_r(\mathcal{A}) & \xrightarrow{F_X} & \mathcal{C}(k \text{ mod}) \end{array}$$

commutes for every $X \in \mathcal{A}$ and for any choice of $n \in \mathbb{Z}_{\geq 0}$.

Proof. Pick a morphism $\varphi : A_\bullet \rightarrow B_\bullet \in \text{Mor}_r(\mathcal{C}_r^n(\mathcal{A}))$. Then

$$\begin{aligned} \text{Cone}(\tilde{F}_X(\varphi))_i &= \text{Hom}(X, A_{i-1}) \oplus F_X(B_i) = \\ &= \begin{cases} \text{Hom}(X, A_{i-1}) \oplus \text{Hom}(X, B_i) & \text{if } i \neq 0 \\ \text{Hom}(X, A_{-1}) \oplus \text{rad}(X, B_0) = \text{rad}(X, B_0) & \text{if } i = 0 \end{cases} \end{aligned}$$

and the differential $d_i : \text{Cone}(\tilde{F}_X(\varphi))_i \rightarrow \text{Cone}(\tilde{F}_X(\varphi))_{i-1}$ is given by

$$d_i = \begin{bmatrix} -d_{i-1}^A \circ - & 0 \\ -\varphi_{i-1} \circ - & d_i^B \circ - \end{bmatrix}.$$

On the other hand, we have

$$F_X(\text{Cone}(\varphi))_i = \begin{cases} \text{Hom}(X, A_{i-1} \oplus B_i) = \text{Hom}(X, A_{i-1}) \oplus \text{Hom}(X, B_i) & \text{if } i \neq 0 \\ \text{rad}(X, A_{-1} \oplus B_0) = \text{rad}(X, B_0) & \text{if } i = 0 \end{cases}$$

and the differential $d'_i : F_X(\text{Cone}(\varphi))_i \rightarrow F_X(\text{Cone}(\varphi))_{i-1}$ is given by

$$d'_i = d_i^{\text{Cone}(\varphi)} \circ - = \begin{bmatrix} -d_{i-1}^A \circ - & 0 \\ -\varphi_{i-1} \circ - & d_i^B \circ - \end{bmatrix}. \quad \square$$

We get a useful criterion for checking whether the cone of a chain map is an n -almost split sequence.

Lemma 2.7 (Criterion for n -almost splitness). *Let $A_\bullet^0 \in \mathcal{C}_r^n(\mathcal{A}_{i_0})$, $A_\bullet^1 \in \mathcal{C}_r^n(\mathcal{A}_{i_0+1})$ for some i_0 . Let $\varphi : A_\bullet^0 \rightarrow A_\bullet^1$ be a chain map. Then the following are equivalent:*

- (1) $\text{Cone}(\varphi)$ is an n -almost split sequence.
- (2) A_n^0 and A_0^1 are indecomposable, and $\tilde{F}_X(\varphi)$ is a quasi-isomorphism for every $X \in \mathcal{A}$.

Proof. (1) \Rightarrow (2). Suppose that $\text{Cone}(\varphi)$ is n -almost split. Then by definition $A_n^0 = \text{Cone}(\varphi)_{n+1}$ and $A_0^1 = \text{Cone}(\varphi)_0$ are indecomposable and $F_X(\text{Cone}(\varphi))$ is exact for every $X \in \mathcal{A}$. By Lemma 2.6 we know that $F_X(\text{Cone}(\varphi)) = \text{Cone}(\tilde{F}_X(\varphi))$, and by Lemma 2.2 exactness of $\text{Cone}(\tilde{F}_X(\varphi))$ implies that $\tilde{F}_X(\varphi)$ is a quasi-isomorphism.

(2) \Rightarrow (1). If $\tilde{F}_X(\varphi)$ is a quasi-isomorphism for every $X \in \mathcal{A}$, then by Lemma 2.2 we know that $\text{Cone}(\tilde{F}_X(\varphi))$ is exact, so by Lemma 2.6 we get that $F_X(\text{Cone}(\varphi))$ is exact for every $X \in \mathcal{A}$. Then by observing that $\text{Cone}(\varphi)_{n+1} = A_n^0$ and $\text{Cone}(\varphi)_0 = A_0^1$ are indecomposable, we can conclude that $\text{Cone}(\varphi)$ is n -almost split. \square

3. Tensor product of mapping cones

3.1. Construction

All tensor products are understood to be over k , even when it is not specified to simplify the notation. The tensor product bifunctor

$$- \otimes - : \text{mod } k \times \text{mod } k \rightarrow \text{mod } k$$

induces (for a general construction, see [4, IV.4,5]) a bifunctor

$$- \otimes^T - : \mathcal{C}(\text{mod } k) \times \mathcal{C}(\text{mod } k) \rightarrow \mathcal{C}(\text{mod } k)$$

(for clarity, we use the symbol \otimes for modules and \otimes^T for complexes). Moreover, since the tensor product defines a bifunctor

$$- \otimes - : \text{mod } A \times \text{mod } B \rightarrow \text{mod}(A \otimes B)$$

we can consider \otimes^T as a bifunctor

$$- \otimes^T - : \mathcal{C}(\text{mod } A) \times \mathcal{C}(\text{mod } B) \rightarrow \mathcal{C}(\text{mod } A \otimes B).$$

For convenience, we give the explicit formulas: on objects, we have

$$(A \otimes^T B)_m = \bigoplus_{j \in \mathbb{Z}} A_j \otimes B_{m-j}$$

with differential d given on an element $v \otimes w \in A_j \otimes B_{m-j}$ by

$$d_m(v \otimes w) = d_j^A(v) \otimes w + (-1)^j v \otimes d_{m-j}^B(w).$$

On morphisms, if $\varphi : A_\bullet^0 \rightarrow A_\bullet^1$ and $\psi : B_\bullet^0 \rightarrow B_\bullet^1$ are chain maps, then

$$(\varphi \otimes^T \psi)_m = \bigoplus_{j \in \mathbb{Z}} \varphi_j \otimes \psi_{m-j} : \bigoplus_{j \in \mathbb{Z}} A_j^0 \otimes B_{m-j}^0 \rightarrow \bigoplus_{j \in \mathbb{Z}} A_j^1 \otimes B_{m-j}^1.$$

Lemma 3.1. *Let A, B be finite-dimensional k -algebras, let \mathcal{A}, \mathcal{B} be subcategories of $\text{mod } A$ and $\text{mod } B$ respectively, and let $\varphi : A_\bullet^0 \rightarrow A_\bullet^1$ and $\psi : B_\bullet^0 \rightarrow B_\bullet^1$ be objects of $\text{Mor}(\mathcal{C}(\mathcal{A}))$ and $\text{Mor}(\mathcal{C}(\mathcal{B}))$ respectively. Suppose that both φ and ψ are quasi-isomorphisms. Then $\varphi \otimes^T \psi$ is a quasi-isomorphism.*

Proof. This follows from the Künneth formula over a field (see [4, VI.3.3.1]). That is, for complexes A_\bullet and B_\bullet there is for every n a functorial isomorphism

$$H_n(A_\bullet \otimes^T B_\bullet) \cong \bigoplus_{i+j=n} H_i(A_\bullet) \otimes H_j(B_\bullet).$$

In our case, this gives for every n an isomorphism

$$H_n(\varphi \otimes^T \psi) \cong (H_i(\varphi) \otimes H_j(\psi))_{i+j=n}.$$

Since φ and ψ are quasi-isomorphisms, the right-hand side is an isomorphism, hence $\varphi \otimes^T \psi$ is a quasi-isomorphism. \square

3.2. Preparation

We now focus on the tensor product of homogeneous algebras. In this case the tensor product behaves well (recall that we are assuming k to be perfect). More precisely, we have the following classical result:

Proposition 3.2. *Let A, B be finite-dimensional k -algebras. Then*

$$\text{gl. dim}(A \otimes_k B) = \text{gl. dim}(A) + \text{gl. dim}(B).$$

Proof. Using a result by Auslander ([3, Theorem 16]), we can assume that A and B are semisimple. Then the claim is a special case of [10, Corollary 5.7]. \square

In our setting, perfectness of the ground field and homogeneity are enough to guarantee that higher representation finiteness is preserved by tensor products:

Theorem 3.3. *Let A be an n -representation finite k -algebra, and let B be an m -representation finite k -algebra. If A and B are l -homogeneous, then the algebra $A \otimes_k B$ is $(n + m)$ -representation finite, l -homogeneous. Moreover, an $(n + m)$ -cluster tilting module for $A \otimes_k B$ is given by*

$$M_{A \otimes B} = \bigoplus_{i=0}^{l-1} \tau_n^{-i} A \otimes \tau_m^{-i} B.$$

Proof. See [6, 1.5]. \square

Proposition 3.4. *Let A and B be two finite-dimensional k -algebras. Let $M, N \in \text{mod } A$ and $M', N' \in \text{mod } B$. Then the canonical map*

$$\text{Hom}_A(M, N) \otimes_k \text{Hom}_B(M', N') \rightarrow \text{Hom}_{A \otimes_k B}(M \otimes_k M', N \otimes_k N')$$

given by $f \otimes g \mapsto f \otimes g$ is an isomorphism of k -vector spaces.

Proof. See Proposition XI.1.2.3 and Theorem XI.3.1 in [4]. \square

We will use the above identification quite freely from now on. We need two more lemmas:

Lemma 3.5. *Let R and S be finite-dimensional k -algebras. Then we have*

$$\text{rad}(R) \otimes_k S + R \otimes_k \text{rad}(S) = \text{rad}(R \otimes_k S)$$

as ideals of $R \otimes_k S$.

Proof. This is [10, Corollary 5.8], combined with the observation that for finite-dimensional algebras the Baer radical and the Jacobson radical coincide (see [11, Proposition 10.27]). \square

Lemma 3.6. *Let A and B be two finite-dimensional k -algebras. Let $M, N \in \text{mod } A$ and $M', N' \in \text{mod } B$. Then we have*

$$\text{rad}(M, N) \otimes \text{Hom}(M', N') + \text{Hom}(M, N) \otimes \text{rad}(M', N') = \text{rad}(M \otimes M', N \otimes N')$$

as subspaces of $\text{Hom}(M \otimes M', N \otimes N')$. Moreover, there is an exact sequence

$$0 \longrightarrow \text{rad}(M) \otimes \text{rad}(M') \xrightarrow{\begin{bmatrix} \alpha \\ -\alpha \end{bmatrix}} \begin{array}{c} \text{rad}(M) \otimes \text{End}(M') \\ \oplus \\ \text{End}(M) \otimes \text{rad}(M') \end{array} \xrightarrow{\begin{bmatrix} \alpha & \alpha \end{bmatrix}} \text{rad}(M \otimes M') \longrightarrow 0$$

where

$$\alpha : f \otimes g \mapsto f \otimes g.$$

Proof. Let $R = \text{End}_A(M \oplus N)$ and $S = \text{End}_B(M' \oplus N')$. By Proposition 3.4 we have

$$R \otimes S \cong \text{End}_{A \otimes B}((M \oplus N) \otimes (M' \oplus N')).$$

Let $p, q \in R$ be the projections onto M, N respectively, and let $p', q' \in S$ be the projections onto M', N' respectively. Then we have

$$(q \otimes q')(\text{rad}(R \otimes S))(p \otimes p') = \text{rad}(M \otimes M', N \otimes N').$$

By Lemma 3.5,

$$\text{rad}(R \otimes S) = \text{rad}(R) \otimes S + R \otimes \text{rad}(S)$$

so that

$$\begin{aligned} \text{rad}(M \otimes M', N \otimes N') &= (q \otimes q')(\text{rad}(R) \otimes S + R \otimes \text{rad}(S))(p \otimes p') = \\ &= \text{rad}(M, N) \otimes \text{Hom}(M', N') + \text{Hom}(M, N) \otimes \text{rad}(M', N'), \end{aligned}$$

which proves the first claim. Moreover, in the case $M = N, M' = N'$ we easily get the exact sequence by looking at the kernel of the map

$$[\alpha \quad \alpha] : \begin{array}{c} \text{rad}(M) \otimes \text{End}(M') \\ \oplus \\ \text{End}(M) \otimes \text{rad}(M') \end{array} \longrightarrow \text{rad}(M \otimes M'). \quad \square$$

3.3. Proof of main result

We are ready to prove [Theorem 1.1](#):

Proof of Theorem 1.1. We fix $\varphi : A^\bullet_0 \rightarrow A^\bullet_1$ and $\psi : B^\bullet_0 \rightarrow B^\bullet_1$. By definition $C_\bullet = \text{Cone}(\varphi \otimes^T \psi)$ is a complex bounded between 0 and $n + m + 1$, and it is exact by [Lemma 2.2](#) and [Lemma 3.1](#). It follows from [Lemma 3.6](#) that

$$(\varphi \otimes^T \psi)_i \in \text{rad}((A^\bullet_0 \otimes^T B^\bullet_0)_i, (A^\bullet_1 \otimes^T B^\bullet_1)_i)$$

for every i , and so $C_\bullet \in \mathcal{C}_r(\mathcal{A} \otimes \mathcal{B})$. Fix an indecomposable $M \otimes N \in \mathcal{A} \otimes \mathcal{B}$. We can consider the maps

$$\tilde{F}_M(\varphi) \otimes^T \tilde{F}_N(\psi) : \text{Hom}(M, A^\bullet_0) \otimes^T \text{Hom}(N, B^\bullet_0) \rightarrow F_M(A^\bullet_1) \otimes^T F_N(B^\bullet_1)$$

and

$$\tilde{F}_{M \otimes N}(\varphi \otimes^T \psi) : \text{Hom}(M \otimes N, A^\bullet_0 \otimes^T B^\bullet_0) \rightarrow F_{M \otimes N}(A^\bullet_1 \otimes^T B^\bullet_1).$$

By [Lemma 3.6](#), the map

$$\iota : \text{Hom}(M, A^\bullet_1) \otimes^T \text{Hom}(N, B^\bullet_1) \rightarrow \text{Hom}(M \otimes N, A^\bullet_1 \otimes^T B^\bullet_1), \quad f \otimes g \mapsto f \otimes g$$

induces a monomorphism

$$\iota' : F_M(A^\bullet_1) \otimes^T F_N(B^\bullet_1) \rightarrow F_{M \otimes N}(A^\bullet_1 \otimes^T B^\bullet_1)$$

so there is a commutative diagram

$$\begin{array}{ccc} \text{Hom}(M, A^\bullet_0) \otimes^T \text{Hom}(N, B^\bullet_0) & \xrightarrow{\iota} & \text{Hom}(M \otimes N, A^\bullet_0 \otimes^T B^\bullet_0) \\ \tilde{F}_M(\varphi) \otimes^T \tilde{F}_N(\psi) \downarrow & & \downarrow \tilde{F}_{M \otimes N}(\varphi \otimes^T \psi) \\ F_M(A^\bullet_1) \otimes^T F_N(B^\bullet_1) & \xrightarrow{\iota'} & F_{M \otimes N}(A^\bullet_1 \otimes^T B^\bullet_1). \end{array}$$

By [Proposition 3.4](#), the map ι is an isomorphism. Moreover, since $\text{Cone}(\varphi)$ is n -almost split, it follows by [Lemma 2.7](#) that $\tilde{F}_M(\varphi)$ is a quasi-isomorphism, and similarly $\tilde{F}_N(\psi)$ is a quasi-isomorphism because $\text{Cone}(\psi)$ is m -almost split. Then by [Lemma 3.1](#) it follows that $\tilde{F}_M(\varphi) \otimes^T \tilde{F}_N(\psi)$ is a quasi-isomorphism. Again by [Lemma 2.7](#), the claim that C_\bullet is $(m + n)$ -almost split will follow if we prove that $\tilde{F}_{M \otimes N}(\varphi \otimes^T \psi)$ is a quasi-isomorphism (since $M \otimes N$ is an arbitrary indecomposable). By the above observations, it is enough to show that ι' is a quasi-isomorphism. This is in turn equivalent to $\text{coker } \iota'$ being exact, which is what we prove. We claim that we have

$$\text{coker } \iota' = F_M(A_\bullet^1) \otimes_k S(N, B_0^1) \oplus S(M, A_0^1) \otimes_k F_N(B_\bullet^1). \tag{1}$$

Assume that this claim holds, and let us prove the theorem. Notice that $S(N, B_0^1) = 0$ unless $N \cong B_0^1$ since B_0^1 and N are indecomposable. Suppose that $N \cong B_0^1$. Then in particular $N \in \text{add } \tau_m^{-(i+1)} B$ and so $M \in \text{add } \tau_n^{-(i+1)} A$ since $M \otimes N \in \mathcal{A} \otimes \mathcal{B}$ (see [Theorem 3.3](#)). Then by [Proposition 2.1](#) we get

$$\text{Hom}(M, A_\bullet^0) = 0.$$

In this case $F_M(A_\bullet^1) \cong \text{Cone}(\tilde{F}_M(\varphi))$ which by [Lemma 2.7](#) is exact if and only if $\text{Cone}(\varphi)$ is n -almost split, which we are assuming. Tensoring over k is exact, so it follows that the first summand in (1) is exact. By symmetry, the second summand is exact as well and we are done.

It remains to prove the equality (1). Call $D_\bullet = F_M(A_\bullet^1) \otimes^T F_N(B_\bullet^1)$. We have that

$$D_p = \bigoplus_{i+j=p} F_M(A_\bullet^1)_i \otimes F_N(B_\bullet^1)_j$$

and we are interested in computing the cokernels of the maps

$$\iota'_p : D_p \rightarrow F_{M \otimes N} (A_\bullet^1 \otimes^T B_\bullet^1)_p.$$

We proceed by first considering the case $p \neq 0$. Then the codomain of ι'_p is

$$\text{Hom} \left(M \otimes N, \bigoplus_{i+j=p} A_i^1 \otimes B_j^1 \right) \cong \bigoplus_{i+j=p} \text{Hom}(M, A_i^1) \otimes \text{Hom}(N, B_j^1)$$

and ι'_p is just the canonical diagonal immersion with components

$$\iota'_{ij} : F_M(A_\bullet^1)_i \otimes F_N(B_\bullet^1)_j \rightarrow \text{Hom}(M, A_i^1) \otimes \text{Hom}(N, B_j^1)$$

given by $f \otimes g \mapsto f \otimes g$. In particular, ι'_{ij} is the identity unless either $i = 0$ and $M \cong A_0^1$ or $j = 0$ and $N \cong B_0^1$. It follows that

$$\text{coker } \iota'_p = \bigoplus_{i+j=p} \text{coker } \iota'_{ij} = \text{coker } \iota'_{0p} \oplus \text{coker } \iota'_{p0}.$$

Let us then suppose $N \cong B_0^1$, and focus on terms of the form $\text{coker } \iota'_{p0}$, where

$$\iota'_{p0} : \text{Hom}(M, A_p^1) \otimes \text{rad}(B_0^1) \rightarrow \text{Hom}(M \otimes B_0^1, A_p^1 \otimes B_0^1).$$

We know by [Proposition 3.4](#) that the right-hand side is canonically isomorphic to $\text{Hom}(M, A_p^1) \otimes \text{End}(B_0^1)$, so from the exact sequence

$$0 \longrightarrow \text{rad}(B_0^1) \longrightarrow \text{End}(B_0^1) \longrightarrow S(B_0^1) \longrightarrow 0$$

we conclude that $\text{coker } \iota'_{p0} = \text{Hom}(M, A_p^1) \otimes S(B_0^1)$. By symmetry we conclude that if $p \neq 0$ then

$$\text{coker } \iota'_p = \text{Hom}(M, A_p^1) \otimes S(N, B_0^1) \oplus S(M, A_0^1) \otimes \text{Hom}(N, B_p^1).$$

Let us analyse the case $p = 0$. Under the identification given by [Proposition 3.4](#), the map

$$\iota'_0 : \text{rad}(M, A_0^1) \otimes \text{rad}(N, B_0^1) \rightarrow \text{rad}(M \otimes N, A_0^1 \otimes B_0^1)$$

is the identity if $M \not\cong A_0^1$ and $N \not\cong B_0^1$, and the inclusion otherwise. If $M \not\cong A_0^1$ and $N \cong B_0^1$, then we are in the same situation as in the previous case, and

$$\text{coker } \iota'_0 = \text{Hom}(M, A_0^1) \otimes S(B_0^1)$$

and similarly for the symmetric case. If both $M \cong A_0^1$ and $N \cong B_0^1$, then we claim that

$$\text{coker } \iota'_0 = \text{rad}(A_0^1) \otimes S(B_0^1) \oplus S(A_0^1) \otimes \text{rad}(B_0^1).$$

Indeed (for simplicity, write $E = A_0^1$ and $F = B_0^1$), in the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \text{rad}(E) \otimes \text{rad}(F) & \xrightarrow{\begin{bmatrix} \alpha \\ -\alpha \end{bmatrix}} & \begin{array}{c} \text{rad}(E) \otimes \text{rad}(F) \\ \oplus \\ \text{rad}(E) \otimes \text{rad}(F) \end{array} & \xrightarrow{\begin{bmatrix} \alpha & \alpha \end{bmatrix}} & \text{rad}(E) \otimes \text{rad}(F) \longrightarrow 0 \\
 & & \parallel & & \downarrow \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} & & \downarrow \iota'_0 \\
 0 & \longrightarrow & \text{rad}(E) \otimes \text{rad}(F) & \xrightarrow{\begin{bmatrix} \alpha \\ -\alpha \end{bmatrix}} & \begin{array}{c} \text{rad}(E) \otimes \text{End}(F) \\ \oplus \\ \text{End}(E) \otimes \text{rad}(F) \end{array} & \xrightarrow{\begin{bmatrix} \alpha & \alpha \end{bmatrix}} & \text{rad}(E \otimes F) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & \begin{array}{c} \text{rad}(E) \otimes S(F) \\ \oplus \\ S(E) \otimes \text{rad}(F) \end{array} & & \text{coker } \iota'_0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

the first row is exact, as well as all the columns (α denotes the canonical map $f \otimes g \mapsto f \otimes g$). The second row is exact by Lemma 3.6. Hence we get an isomorphism

$$\text{rad}(E) \otimes S(F) \oplus S(E) \otimes \text{rad}(F) \cong \text{coker } \iota'_0$$

by the 3×3 lemma. We have shown that

$$\text{coker } \iota'_p = F_M(A_p^1) \otimes S(N, B_0^1) \oplus S(M, A_0^1) \otimes F_N(B_p^1)$$

for every value of $p = 0, \dots, m + n$.

It remains to show that the differentials $\text{coker } \iota'_{p+1} \rightarrow \text{coker } \iota'_p$ are diagonal, to conclude that the direct-sum decomposition of the objects is actually a direct-sum decomposition into the two complexes appearing in equation (1). The only degree where this poses problems is $p = 0$ in the case $M \cong E = A_0^1$, $N \cong F = B_0^1$. For this, consider the following diagram:

$$\begin{array}{ccc}
 \begin{array}{c} \text{Hom}(E, A_1^1) \otimes \text{rad}(F) \\ \oplus \\ \text{rad}(E) \otimes \text{Hom}(F, B_1^1) \end{array} & \xrightarrow{\beta} & \text{rad}(E) \otimes \text{rad}(F) \\
 \downarrow \iota'_1 & & \downarrow \iota'_0 \\
 \begin{array}{c} \text{Hom}(E, A_1^1) \otimes \text{End}(F) \\ \oplus \\ \text{End}(E) \otimes \text{Hom}(F, B_1^1) \end{array} & \xrightarrow{\beta} & \text{rad}(E \otimes F) \\
 \downarrow & & \downarrow \\
 \begin{array}{c} \text{Hom}(E, A_1^1) \otimes S(F) \\ \oplus \\ S(E) \otimes \text{Hom}(F, B_1^1) \end{array} & \xrightarrow{\quad} & \text{coker } \iota'_0
 \end{array}$$

where the horizontal maps are induced by

$$\beta = [(d_1^A \circ -) \otimes \text{id}, \text{id} \otimes (d_1^B \circ -)],$$

which is the last map appearing in the sequence $F_E(A_\bullet^1) \otimes^T F_F(B_\bullet^1)$. The map β factors as

$$\beta = [\alpha \quad \alpha] \begin{bmatrix} (d_1^A \circ -) \otimes \text{id} & 0 \\ 0 & \text{id} \otimes (d_1^B \circ -) \end{bmatrix}$$

hence the diagram above can be completed to a diagram

$$\begin{array}{ccccc}
 \begin{array}{c} \text{Hom}(E, A_1^1) \otimes \text{rad}(F) \\ \oplus \\ \text{rad}(E) \otimes \text{Hom}(F, B_1^1) \end{array} & \xrightarrow{\quad} & \begin{array}{c} \text{rad}(E) \otimes \text{rad}(F) \\ \oplus \\ \text{rad}(E) \otimes \text{rad}(F) \end{array} & \xrightarrow{[\alpha \quad \alpha]} & \text{rad}(E) \otimes \text{rad}(F) \\
 \downarrow \iota'_1 & & \downarrow & & \downarrow \iota'_0 \\
 \begin{array}{c} \text{Hom}(E, A_1^1) \otimes \text{End}(F) \\ \oplus \\ \text{End}(E) \otimes \text{Hom}(F, B_1^1) \end{array} & \xrightarrow{\quad} & \begin{array}{c} \text{rad}(E) \otimes \text{End}(F) \\ \oplus \\ \text{End}(E) \otimes \text{rad}(F) \end{array} & \xrightarrow{[\alpha \quad \alpha]} & \text{rad}(E \otimes F) \\
 \downarrow & & \downarrow & & \downarrow \\
 \begin{array}{c} \text{Hom}(E, A_1^1) \otimes S(F) \\ \oplus \\ S(E) \otimes \text{Hom}(F, B_1^1) \end{array} & \xrightarrow{\quad} & \begin{array}{c} \text{rad}(E) \otimes S(F) \\ \oplus \\ S(E) \otimes \text{rad}(F) \end{array} & \xrightarrow{\cong} & \text{coker } \iota'_0
 \end{array}$$

where the horizontal maps on the left-hand side are diagonal. Hence the induced map

$$\begin{array}{ccc}
 \text{Hom}(E, A_1^1) \otimes S(F) & & \\
 \oplus & \xrightarrow{\quad} & \text{coker } \iota'_0 \\
 S(E) \otimes \text{Hom}(F, B_1^1) & &
 \end{array}$$

factors through the diagonal map

$$\begin{bmatrix} (d_1^A \circ -) \otimes \text{id}_{S(F)} & 0 \\ 0 & \text{id}_{S(E)} \otimes (d_1^B \circ -) \end{bmatrix} : \begin{array}{ccc} \text{Hom}(E, A_1^1) \otimes S(F) & \xrightarrow{\quad} & \text{rad}(E) \otimes S(F) \\ \oplus & & \oplus \\ S(E) \otimes \text{Hom}(F, B_1^1) & & S(E) \otimes \text{rad}(F) \end{array}$$

and we are done. \square

4. Examples

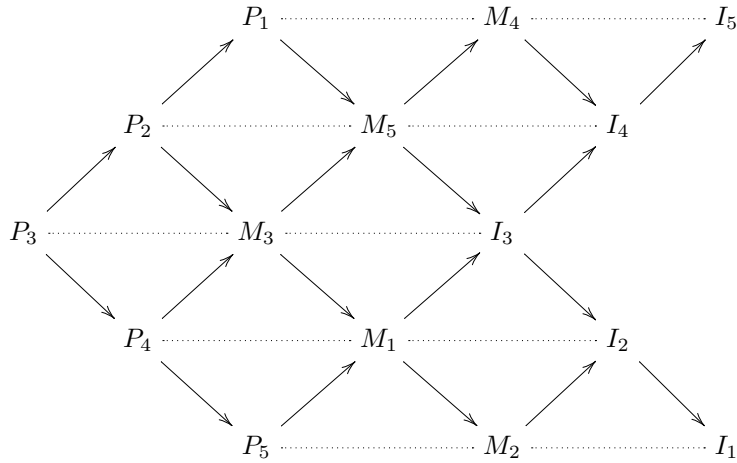
As an example, consider the quiver

$$1 \longrightarrow 2 \longrightarrow 3 \longleftarrow 4 \longleftarrow 5$$

and the corresponding path algebra $A = kQ$. Thus A is 3-homogeneous, (1-) representation finite (see [2,6]). We want to consider the algebra $B = A \otimes A$, which is then 3-homogeneous, 2-representation finite. There are 15 nonisomorphic indecomposables in $\text{mod } A$, which have the following dimension vectors:

$$\begin{aligned} P_1 &: (11100) & M_1 &: (01111) & I_1 &: (10000) \\ P_2 &: (01100) & M_2 &: (01000) & I_2 &: (11000) \\ P_3 &: (00100) & M_3 &: (01110) & I_3 &: (11111) \\ P_4 &: (00110) & M_4 &: (00010) & I_4 &: (00011) \\ P_5 &: (00111) & M_5 &: (11110) & I_5 &: (00001). \end{aligned}$$

The Auslander–Reiten quiver of A is the following:



where the dotted lines represent τ_1^- .

Inside $\text{mod } B$ we have the 2-cluster tilting subcategory $\mathcal{C} = \text{add } M$, where

$$M = \bigoplus_{1 \leq i, j \leq 5} P_i \otimes P_j \oplus \bigoplus_{1 \leq i, j \leq 5} M_i \otimes M_j \oplus \bigoplus_{1 \leq i, j \leq 5} I_i \otimes I_j.$$

Let us consider for instance the (1-)almost split sequences

$$C_\bullet = 0 \longrightarrow P_2 \xrightarrow{\begin{bmatrix} a \\ b \end{bmatrix}} P_1 \oplus M_3 \xrightarrow{\begin{bmatrix} c & d \end{bmatrix}} M_5 \longrightarrow 0$$

and

$$D_\bullet = 0 \longrightarrow P_5 \xrightarrow{e} M_1 \xrightarrow{f} M_2 \longrightarrow 0$$

in $\text{mod } A$. Notice that both these sequences start in slice 0. The sequence C_\bullet is isomorphic to the cone of

$$\begin{array}{ccccccc} \dots & \longrightarrow & 0 & \longrightarrow & P_2 & \xrightarrow{-a} & P_1 & \longrightarrow & 0 & \longrightarrow & \dots \\ & & \downarrow & & \downarrow -b & & \downarrow -c & & \downarrow & & \\ \dots & \longrightarrow & 0 & \longrightarrow & M_3 & \xrightarrow{d} & M_5 & \longrightarrow & 0 & \longrightarrow & \dots \end{array}$$

and D_\bullet is isomorphic to the cone of

$$\begin{array}{ccccccccc}
 \cdots & \longrightarrow & 0 & \longrightarrow & P_5 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 \cdots & \longrightarrow & 0 & \longrightarrow & M_1 & \xrightarrow{f} & M_2 & \longrightarrow & 0 & \longrightarrow & \cdots
 \end{array}$$

where these diagrams should be seen as morphisms φ, ψ of chain complexes. Then we can construct the morphism $\varphi \otimes^T \psi$:

$$\begin{array}{ccccccccc}
 \cdots & \longrightarrow & 0 & \longrightarrow & P_2 \otimes P_5 & \xrightarrow{-a \otimes 1} & P_1 \otimes P_5 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 \cdots & \longrightarrow & 0 & \longrightarrow & M_3 \otimes M_1 & \xrightarrow{\begin{bmatrix} -1 \otimes f \\ d \otimes 1 \end{bmatrix}} & M_3 \otimes M_2 \oplus_{M_5 \otimes M_1} & \xrightarrow{\begin{bmatrix} 0 & -1 \otimes f \\ c \otimes e & d \otimes 1 \end{bmatrix}} & M_5 \otimes M_2 & \longrightarrow & 0 & \longrightarrow & \cdots
 \end{array}$$

The cone $E_\bullet = \text{Cone}(\varphi \otimes^T \psi)$ is then the sequence

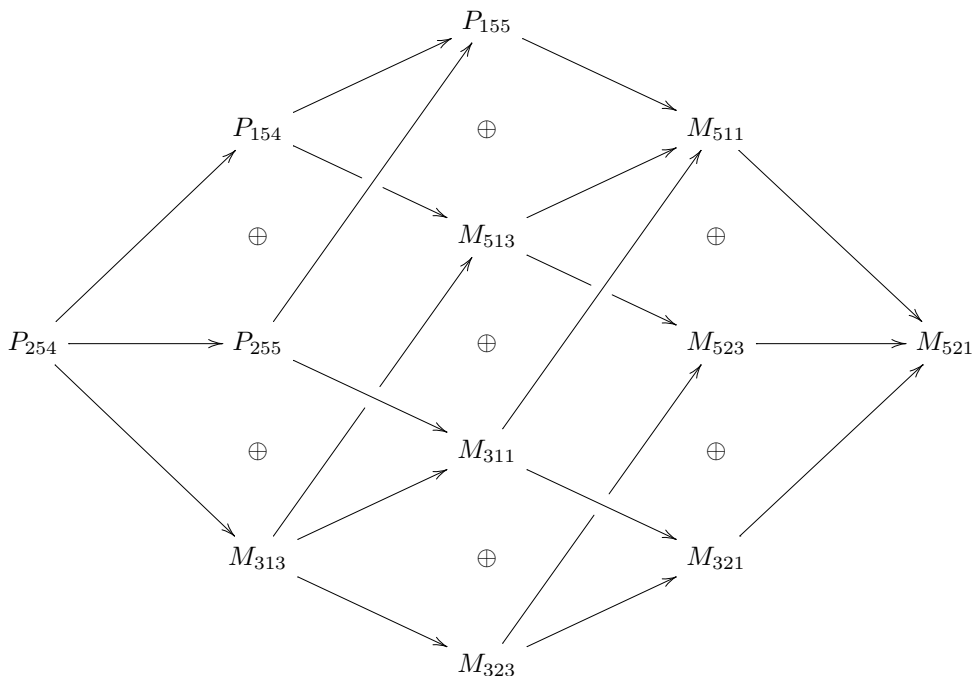
$$0 \longrightarrow P_2 \otimes P_5 \xrightarrow{\begin{bmatrix} a \otimes 1 \\ -b \otimes e \end{bmatrix}} M_3 \otimes M_1 \oplus_{P_1 \otimes P_5} \xrightarrow{\begin{bmatrix} 0 & -1 \otimes f \\ -c \otimes e & d \otimes 1 \end{bmatrix}} M_3 \otimes M_2 \oplus_{M_5 \otimes M_1} \xrightarrow{\begin{bmatrix} d \otimes 1 & 1 \otimes f \end{bmatrix}} M_5 \otimes M_2 \longrightarrow 0$$

which is 2-almost split in \mathcal{C} by Theorem 1.1.

Now we can go further, and consider the algebra $B \otimes A$, which is then 3-homogeneous, 3-representation finite. Let us write for simplicity $P_{abc} = P_a \otimes P_b \otimes P_c$ and $M_{abc} = M_a \otimes M_b \otimes M_c$. We look at the 3-almost split sequence starting in P_{254} , which is obtained from E_\bullet together with the sequence

$$0 \longrightarrow P_4 \longrightarrow P_5 \oplus M_3 \longrightarrow M_1 \longrightarrow 0$$

in mod A . By applying the formula we get the sequence



where each arrow is the natural morphism up to sign.

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