



Tensor products of n -complete algebras

Andrea Pasquali

Dept. of Mathematics, Uppsala University, P.O. Box 480, 751 06 Uppsala, Sweden



ARTICLE INFO

Article history:

Received 12 January 2017

Received in revised form 9 October 2018

Available online 16 November 2018

Communicated by S. Koenig

MSC:

16G70; 16G10; 16E10

ABSTRACT

If A and B are n - and m -representation finite k -algebras, then their tensor product $\Lambda = A \otimes_k B$ is not in general $(n + m)$ -representation finite. However, we prove that if A and B are acyclic and satisfy the weaker assumption of n - and m -completeness, then Λ is $(n + m)$ -complete. This mirrors the fact that taking higher Auslander algebra does not preserve d -representation finiteness in general, but it does preserve d -completeness. As a corollary, we get the necessary condition for Λ to be $(n + m)$ -representation finite which was found by Herschend and Iyama by using a certain twisted fractionally Calabi–Yau property.

© 2018 Elsevier B.V. All rights reserved.

1. Introduction

Higher Auslander–Reiten theory was developed in a series of papers [10], [9], [11] as a tool to study module categories of finite-dimensional algebras. The idea is to replace all the homological notions in classical Auslander–Reiten theory with higher-dimensional analogs. Some early results can be found in [8], [5]. This approach has been fruitful in the context of noncommutative algebraic geometry, see for instance [1], [7], [6]. Higher Auslander–Reiten theory is also deeply tied with d -homological algebra ([3], [13], [15]). A presentation of the theory from this point of view can be found in [14].

In this setting, d -representation finite algebras were introduced in [12] as a generalisation of hereditary representation finite algebras. They are algebras of global dimension at most d that have a d -cluster tilting module M . The category $\text{add } M$ has nice homological properties and behaves in many ways like the module category of a hereditary representation finite algebra. While classification of d -representation finite algebras seems far from being achieved, it makes sense to look for examples, and to try to understand how d -representation finiteness behaves with respect to reasonable operations. Notice that in this setting we have more freedom than in the hereditary case, since we are allowed to increase the global dimension and still fall within the scope of the theory.

E-mail address: andrea.pasquali@math.uu.se.

For instance, in [12] Iyama investigates whether the endomorphism algebra of the d -cluster tilting module (called the higher Auslander algebra) is $(d + 1)$ -representation finite. This turns out to be false in general, but a necessary and sufficient condition is given: the only case where it is true is within the tower of iterated higher Auslander algebras of the upper triangular matrix algebra, so this construction gives only a specific family of examples. On the other hand, in the same paper the weaker notion of d -complete algebra is introduced and studied. A d -complete algebra is an algebra of global dimension at most d that has a module which is d -cluster tilting in a suitable exact subcategory of the module category. It turns out that this weaker notion is preserved under taking higher Auslander algebras, thereby producing many examples of d -complete algebras for any d .

Another operation one might investigate is that of taking tensor products over the base field k . Indeed, if k is perfect then $\text{gl. dim } A \otimes_k B = \text{gl. dim } A + \text{gl. dim } B$, so it makes sense to ask whether the tensor product of an n - and an m -representation finite algebras is $(n + m)$ -representation finite. This is false in general, and in [4] Herschend and Iyama give a necessary and sufficient condition (l -homogeneity) for it to be true.

In this paper we prove that the same weaker notion of d -completeness which is used in [12] is preserved under tensor products, under the assumption of acyclicity. Namely, if A is n -complete and acyclic and B is m -complete and acyclic, then $A \otimes_k B$ is $(n + m)$ -complete and acyclic. If we assume that A and B are l -homogeneous, we recover the result by Herschend and Iyama. This gives a new way of producing d -complete algebras for any d .

The proof we give is structured as follows. We prove that over the tensor product there are $(n + m)$ -almost split sequences (using the same construction as in [16]), and moreover that injective modules have source sequences. Then we use these sequences, combined with the assumption of acyclicity, to prove that the module T in the definition of $(n + m)$ -completeness is tilting. By [12, Theorem 2.2(b)], the existence of the above sequences in T^\perp is equivalent to M being $(n + m)$ -cluster tilting in T^\perp , which is the key point of $(n + m)$ -completeness.

In Section 2 we lay down notation, conventions, and preliminary definitions. Section 3 contains the statement of our main result. Section 4 contains the results about d -almost split sequences and tensor products which we want to use. Section 5 is dedicated to proving the main theorem, which amounts to checking that the tensor product satisfies the defining properties of $(n + m)$ -complete algebras. In Section 6 we present some examples.

2. Notation and conventions

Throughout this paper, k denotes a fixed perfect field. All algebras are associative, unital, and finite dimensional over k . For an algebra Λ , $\text{mod } \Lambda$ (respectively $\Lambda \text{ mod}$) denotes the category of finitely generated right (left) Λ -modules. We denote by D the duality $D = \text{Hom}_k(-, k)$ between $\text{mod } \Lambda$ and $\Lambda \text{ mod}$ (in both directions). Subcategories are always assumed to be full and closed under isomorphisms, finite direct sums and summands. For $M \in \text{mod } \Lambda$, we denote by $\text{add } M$ the subcategory of $\text{mod } \Lambda$ whose objects are all modules isomorphic to finite direct sums of summands of M . We write $\text{rad}_\Lambda(-, -)$ for the subfunctor of $\text{Hom}_\Lambda(-, -)$ defined by

$$\text{rad}_\Lambda(X, Y) = \{f \in \text{Hom}_\Lambda(X, Y) \mid \text{id}_X - g \circ f \text{ is invertible } \forall g \in \text{Hom}_\Lambda(Y, X)\}.$$

Moreover, for $X, Y \in \text{mod } \Lambda$, we write $\text{top}_\Lambda(X, Y) = \text{Hom}_\Lambda(X, Y) / \text{rad}_\Lambda(X, Y)$. We often write Hom instead of Hom_Λ and similarly for rad and top when the context allows it. We denote by $\mathcal{D}^b(\Lambda)$ the bounded derived category of $\text{mod } \Lambda$. For a subcategory \mathcal{C} of $\mathcal{D}^b(\Lambda)$, we denote by $\text{thick } \mathcal{C}$ the smallest triangulated subcategory of $\mathcal{D}^b(\Lambda)$ containing \mathcal{C} . If $\mathcal{C} = \text{add } M$ for some $M \in \text{mod } \Lambda \subseteq \mathcal{D}^b(\Lambda)$, we write $\text{thick } M = \text{thick}(\text{add } M)$. All tensor products are over k , even when the specification is omitted to simplify the notation.

Throughout this section, let $\text{gl. dim } \Lambda \leq d$. Then we can define the *higher Auslander–Reiten translations* by

$$\begin{aligned} \tau_d &= D \text{Ext}_\Lambda^d(-, \Lambda) : \text{mod } \Lambda \rightarrow \text{mod } \Lambda \\ \tau_d^- &= \text{Ext}_{\Lambda^{op}}^d(D-, \Lambda) : \text{mod } \Lambda \rightarrow \text{mod } \Lambda. \end{aligned}$$

We are interested in categories associated to tilting modules.

Definition 2.1. A Λ -module T is *tilting* if the following conditions are satisfied:

- (1) $\text{Ext}^i(T, T) = 0$ for all $i \neq 0$,
- (2) there is an exact sequence $0 \rightarrow \Lambda \rightarrow T_0 \rightarrow \dots \rightarrow T_m \rightarrow 0$ for some m with $T_i \in \text{add } T$ for all i .

The second condition in the definition can be replaced by

$$\text{thick } T = \mathcal{D}^b(\Lambda).$$

For a tilting module T , we have an exact subcategory of $\text{mod } \Lambda$

$$T^\perp = \{X \in \text{mod } \Lambda \mid \text{Ext}^i(T, X) = 0 \text{ for every } i \neq 0\}$$

We are interested in d -cluster tilting subcategories of T^\perp .

Definition 2.2. Let T be a tilting module. A subcategory \mathcal{C} of T^\perp is called *d -cluster tilting* if

$$\begin{aligned} \mathcal{C} &= \{X \in T^\perp \mid \text{Ext}^i(\mathcal{C}, X) = 0 \text{ for every } 0 < i < d\} = \\ &= \{X \in T^\perp \mid \text{Ext}^i(X, \mathcal{C}) = 0 \text{ for every } 0 < i < d\}. \end{aligned}$$

We follow [12, Definition 1.11] and define the following subcategories of $\text{mod } \Lambda$:

- (1) $\mathcal{M} = \mathcal{M}(\Lambda) = \text{add } \{\tau_d^i D\Lambda \mid i \geq 0\}$,
- (2) $\mathcal{P} = \{X \in \mathcal{M} \mid \tau_d X = 0\}$,
- (3) $\mathcal{M}_\mathcal{P} = \{X \in \mathcal{M} \mid X \text{ has no nonzero summands in } \mathcal{P}\}$,
- (4) $\mathcal{M}_I = \{X \in \mathcal{M} \mid X \text{ has no nonzero summands in } \text{add } D\Lambda\}$.

Let T_Λ be a basic module such that $\text{add } T_\Lambda = \mathcal{P}$.

Definition 2.3. An algebra Λ is *d -complete* if the following conditions hold:

- (A_d) T_Λ is a tilting module.
- (B_d) \mathcal{M} is a d -cluster tilting subcategory of T_Λ^\perp .
- (C_d) $\text{Ext}^i(\mathcal{M}_\mathcal{P}, \Lambda) = 0$ for every $0 < i < d$.

Note that condition (A_d) implies that $\tau_d^l = 0$ for large l ([12, Proposition 1.12(d) and 1.3(c)]). Note moreover that if Λ is d -complete then since $\text{gl. dim } \Lambda \leq d$ it follows that $\text{gl. dim } \Lambda \in \{0, d\}$. This is a generalisation of the notion of d -representation finiteness which we use in [16]. Without loss of generality, from now on we assume that Λ is basic. We write T for T_Λ when the context allows it. Then [12, Proposition 1.13] says that “ d -representation finite” is the same as “ d -complete with $T = \Lambda$ ”.

If Λ is d -complete, then for every indecomposable injective I_i there is a unique $l_i \in \mathbb{N}$ such that $\tau_d^{l_i-1}I_i \in \mathcal{P}$, and

$$T_\Lambda = \bigoplus_i \tau_d^{l_i-1}I_i.$$

Definition 2.4 ([4]). Let Λ be a k -algebra of global dimension d such that $\tau_d^l = 0$ for l sufficiently large. We say that Λ is l -homogeneous if $\tau_d^{l-1}D\Lambda = T_\Lambda$.

If Λ is d -complete, this means that $l_i = l$ for every i .

Our main result is proved only for acyclic algebras, let us define what we mean by that. Let $M \in \text{mod } \Lambda$, and let $\mathcal{C} = \text{add } M$. We want to define a preorder on the indecomposable objects $\text{ind } \mathcal{C}$ of \mathcal{C} . For $X, Y \in \text{ind } \mathcal{C}$, we say $X < Y$ if there is a sequence $(X = X_0, X_1, \dots, X_{m+1} = Y)$ for some $m \geq 0$, such that $X_i \in \text{ind } \mathcal{C}$ and $\text{rad}_\Lambda(X_i, X_{i+1}) \neq 0$ for all i . This defines a transitive relation $<$ on $\text{ind } \mathcal{C}$. Notice that we can replace $\text{rad}_\Lambda(X_i, X_{i+1}) \neq 0$ with $\text{rad}_{\mathcal{C}}(X_i, X_{i+1}) \neq 0$ above.

Definition 2.5. The category \mathcal{C} is *directed* if $<$ is antisymmetric, that is if no indecomposable module $X \in \mathcal{C}$ satisfies $X < X$. If $\mathcal{C} = \text{add } M$, we say that M is directed. We call the algebra Λ *acyclic* if Λ_Λ is directed.

3. Main result

We now consider the case where A is n -complete, B is m -complete, and $\Lambda = A \otimes_k B$. Since k is perfect, we have that $\text{gl. dim } \Lambda = \text{gl. dim } A + \text{gl. dim } B$. Moreover, by the Künneth formula we have $\tau_{n+m}X \otimes Y = \tau_n X \otimes \tau_m Y$. Since indecomposable injective Λ -modules are of the form $X \otimes Y$, it follows that all indecomposable modules in \mathcal{M} are of this form. Our main result is the following:

Theorem 3.1. *Let A, B be n - respectively m -complete acyclic k -algebras, with k perfect. Then $A \otimes_k B$ is $(n + m)$ -complete and acyclic.*

Note that as far as the author is aware, there are no known examples of d -complete algebras which are not acyclic (this is Question 5.9 in [7]).

This result can be applied inductively to construct d -complete algebras starting for example from hereditary representation finite algebras and taking tensor products. In this sense, it is similar in spirit to [12, Theorem 1.14 and Corollary 1.16], where Iyama constructs towers of d -complete algebras (with increasing d) by taking iterated higher Auslander algebras. The algebra $A \otimes B$ is almost never $(n + m)$ -representation finite by the characterisation given by Herschend and Iyama in [4]. Our result specialises to their characterisation in the acyclic case:

Corollary 3.2. *Let A, B be n - respectively m -representation finite acyclic k -algebras, with k perfect. Then the following are equivalent:*

- (1) $A \otimes_k B$ is $(n + m)$ -representation finite.
- (2) $\exists l \in \mathbb{N}$ such that A and B are l -homogeneous.

Moreover, in this case $A \otimes_k B$ is also l -homogeneous.

It should be noted that there is a choice involved in the definition we gave of d -completeness, namely that we take \mathcal{M} to be the τ_d -completion of $\text{add } D\Lambda$. We might as well take \mathcal{M} to be the τ_d^- -completion of $\text{add } \Lambda$, and call Λ d -cocomplete if it satisfies the dual conditions to $(A_d), (B_d), (C_d)$. Then Λ is d -complete if and

only if Λ^{op} is d -cocomplete. Notice that d -representation finite is the same as d -complete and d -cocomplete with the same \mathcal{M} . However, if A and B are n - and m -representation finite, then $A \otimes B$ is $(n + m)$ -complete and cocomplete, but in general not with the same \mathcal{M} .

4. Preparation

4.1. d -complete algebras

Following [12], we make some observations about d -complete algebras in general. Fix a finite-dimensional algebra Λ .

Lemma 4.1. *If $\text{gl. dim } \Lambda \leq d$, the following are equivalent:*

- (1) $\text{Ext}^i(\mathcal{M}_P, \Lambda) = 0$ for $0 < i < d$
- (2) $\text{Ext}^i(\mathcal{M}_P, \Lambda) = 0$ for $0 \leq i < d$.

Proof. The only direction to prove follows from [12, Lemma 2.3(b)]. \square

Proposition 4.2. *If Λ is d -complete, then*

$$\text{Hom}(\tau_d^i D\Lambda, \tau_d^j D\Lambda) = 0$$

if $i < j$.

Proof. This follows from [12, Lemma 2.4(e)]. \square

We can define *slices* $\mathcal{S}(i)$ on \mathcal{M} by saying that $\mathcal{S}(i) = \text{add } \tau_d^i D\Lambda$. Thus

$$\mathcal{M} = \bigvee_{i \geq 0} \mathcal{S}(i)$$

(meaning that every object $X \in \mathcal{M}$ can be written uniquely as $X = \bigoplus_{i \geq 0} X_i$ with $X_i \in \mathcal{S}(i)$) and moreover $\text{Hom}(\mathcal{S}(i), \mathcal{S}(j)) = 0$ if $i < j$.

Lemma 4.3. *If Λ is d -complete then τ_d^\pm induce quasi-inverse equivalences $\mathcal{M}_P \leftrightarrow \mathcal{M}_I$.*

Proof. This is [12, Lemma 2.4(b)]. \square

4.2. d -almost split sequences

In the spirit of generalising Auslander–Reiten theory, it is natural to define the higher analog of almost split sequences as follows.

Definition 4.1 (Iyama). A complex with objects in a subcategory \mathcal{C} of $\text{mod } \Lambda$

$$C_d \xrightarrow{f_d} C_{d-1} \xrightarrow{f_{d-1}} C_{d-2} \xrightarrow{f_{d-2}} \dots$$

is a *source sequence (in \mathcal{C}) of C_d* if the following conditions hold:

- (1) $f_i \in \text{rad}(C_i, C_{i-1})$ for all i ,
- (2) the sequence of functors

$$\dots \xrightarrow{-\circ f_{d-2}} \text{Hom}(C_{d-2}, -) \xrightarrow{-\circ f_{d-1}} \text{Hom}(C_{d-1}, -) \xrightarrow{-\circ f_d} \text{rad}(C_d, -) \longrightarrow 0$$

is exact on \mathcal{C} .

Dually we can define *sink sequences*. An exact sequence

$$0 \longrightarrow C_{d+1} \longrightarrow C_{d-1} \longrightarrow \dots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$$

is an *d-almost split sequence* if it is a source sequence of C_{d+1} and a sink sequence of C_0 . We say that such *d-almost split sequence* starts in C_{d+1} and ends in C_0 .

Definition 4.2. We say that $\mathcal{M} = \mathcal{M}(\Lambda)$ has *d-almost split sequences* if for every indecomposable $X \in \mathcal{M}_I$ (respectively $Y \in \mathcal{M}_P$) there is an *d-almost split sequence* in \mathcal{C}

$$0 \rightarrow X \rightarrow C_d \rightarrow \dots \rightarrow C_1 \rightarrow Y \rightarrow 0.$$

In this case we must have $X \cong \tau_d Y, Y \cong \tau_d^- X$. This holds for *d-complete* algebras ([12, Theorem 2.2(a)(i)]):

Theorem 4.4. *If Λ is d-complete, then \mathcal{M} has d-almost split sequences.*

To apply the methods introduced in [16], we need to rephrase Definition 4.1 as follows: for any indecomposable $X \in \mathcal{C}$ we can define a functor F_X on complexes of radical maps by mapping

$$C_\bullet = \dots \xrightarrow{f_{i+1}} C_i \xrightarrow{f_i} \dots \xrightarrow{f_1} C_0 \xrightarrow{f_0} \dots$$

to

$$F_X(C_\bullet) = \dots \xrightarrow{f_{i+1} \circ -} \text{Hom}(X, C_i) \xrightarrow{f_i \circ -} \dots \xrightarrow{f_1 \circ -} \text{rad}(X, C_0) \xrightarrow{f_0 \circ -} \dots$$

(that is, F_X is the subfunctor of $\text{Hom}(X, -)$ given by replacing $\text{Hom}(X, C_0)$ with $\text{rad}(X, C_0)$). Similarly, we can define a subfunctor G_X of the contravariant functor $\text{Hom}(-, X)$ by mapping C_\bullet to

$$G_X(C_\bullet) = \dots \xrightarrow{-\circ f_0} \text{Hom}(C_0, X) \xrightarrow{-\circ f_1} \dots \xrightarrow{-\circ f_{d+1}} \text{rad}(C_{d+1}, X) \xrightarrow{-\circ f_{d+2}} \dots$$

Lemma 4.5. *Let C_\bullet be a complex in \mathcal{C} . Then*

- (1) *If $C_i = 0$ for all $i > d + 1$, then C_\bullet is a sink sequence if and only if $F_X(C_\bullet)$ is exact for every $X \in \mathcal{C}$.*
- (2) *If $C_i = 0$ for all $i < 0$, then C_\bullet is a source sequence if and only if $G_X(C_\bullet)$ is exact for every $X \in \mathcal{C}$.*
- (3) *If $C_i = 0$ for all $i > d + 1$ and $i < 0$, then C_\bullet is d-almost split if and only if $F_X(C_\bullet)$ and $G_X(C_\bullet)$ are exact for every $X \in \mathcal{C}$.*

Proof. Direct check using the definitions. \square

By additivity, in the above Lemma we can replace “every $X \in \mathcal{C}$ ” by “every indecomposable $X \in \mathcal{C}$ ”.

Notice that since d -almost split sequences come from minimal projective resolutions of a functor $\text{rad}(C_0, -)$, they are uniquely determined by C_0 up to isomorphism of complexes. Moreover, we have

Lemma 4.6. *Any map $f_0 : C_0 \rightarrow D_0$ between indecomposables in \mathcal{M}_P induces a map of complexes $f_\bullet : C_\bullet \rightarrow D_\bullet$, where*

$$\begin{aligned}
 C_\bullet = & \quad 0 \longrightarrow C_{d+1} \xrightarrow{g_{d+1}} \cdots \xrightarrow{g_1} C_0 \longrightarrow 0, \\
 D_\bullet = & \quad 0 \longrightarrow D_{d+1} \xrightarrow{h_{d+1}} \cdots \xrightarrow{h_1} D_0 \longrightarrow 0
 \end{aligned}$$

are the d -almost split sequences ending in C_0 and D_0 respectively, if these exist.

Proof. The map $f_0 g_1 : C_1 \rightarrow D_0$ is a radical morphism, and since

$$\text{Hom}(C_1, D_1) \xrightarrow{h_1 \circ -} \text{rad}(C_1, D_0)$$

is surjective, there is a map $f_1 : C_1 \rightarrow D_1$ such that $h_1 f_1 = f_0 g_1$. Now assume we have constructed maps $f_j : C_j \rightarrow D_j$ that make all diagrams commute, for all $0 \leq j < i$ for some $i \geq 2$. We have that

$$\text{Hom}(C_i, D_i) \xrightarrow{h_i \circ -} \text{Hom}(C_i, D_{i-1}) \xrightarrow{h_{i-1} \circ -} \text{Hom}(C_i, D_{i-2})$$

is exact in the middle term by assumption. Since $h_{i-1} f_{i-1} g_i = f_{i-2} g_{i-1} g_i = 0$, we have that $f_{i-1} g_i \in \ker(h_{i-1} \circ -) = \text{im}(h_i \circ -)$, that is there is a map $f_i : C_i \rightarrow D_i$ such that $f_{i-1} g_i = h_i f_i$. The f_i 's we have defined recursively give by construction a map of complexes $f_\bullet : C_\bullet \rightarrow D_\bullet$. \square

The following is a result which appeared in [16] in the setting of d -representation finite algebras, and which can be reformulated in the setting of d -complete algebras.

Theorem 4.7. *Let Λ be d -complete. Let $X \in \mathcal{S}(i)$ with $i > 0$. Then the d -almost split sequence starting in X is isomorphic as a complex to $\text{Cone } \varphi$, where $\varphi : E_\bullet \rightarrow F_\bullet$ is a map of complexes, such that:*

- (1) *All the maps appearing in E_\bullet, F_\bullet , and the components of φ are radical,*
- (2) *$E_j \in \mathcal{S}(i)$ and $F_j \in \mathcal{S}(i - 1)$ for every j .*

Proof. This is shown exactly as in [16, Theorem 2.3]. Namely, one decomposes the modules M_j appearing in the d -almost split sequence starting in X as $M_j = \bigoplus_{i \geq 0} M_{ij}$ with $M_{ij} \in \mathcal{S}(i)$. One checks using Proposition 4.2 that in order for the sequence to be d -almost split, all the M_j must be in $\text{add}(\tau_d^i D \Lambda \oplus \tau_d^{i-1} D \Lambda)$ for some i . Now let $E_j = M_{i,j+1}$ and $F_j = M_{i-1,j}$. Using that $\text{Hom}(\tau_d^{i-1} D \Lambda, \tau_d^i D \Lambda) = 0$ one can choose suitable differentials for E_\bullet and F_\bullet and a morphism $\varphi_\bullet : E_\bullet \rightarrow F_\bullet$ such that $\text{Cone } \varphi$ is the desired sequence. \square

We will need a technical lemma:

Lemma 4.8. *Let*

$$0 \longrightarrow C_{d+1} \xrightarrow{f_{d+1}} C_d \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{f_1} C_0 \longrightarrow 0$$

be a d -almost split sequence. Then for any choice of decomposition of the modules C_i into indecomposables, the corresponding matrices of the maps f_i have no zero column and no zero row.

Proof. We argue by contradiction. Assume f_i has a zero column for $i > 1$. Then there is a complex

$$C_{i+1} \xrightarrow{\begin{bmatrix} f_{i+1}^1 \\ f_{i+1}^2 \end{bmatrix}} C_i^1 \oplus C_i^2 \xrightarrow{\begin{bmatrix} f_i^1 & 0 \end{bmatrix}} C_{i-1}$$

such that

$$\text{Hom}(C_i^2, C_{i+1}) \xrightarrow{\begin{bmatrix} f_{i+1}^1 \circ - \\ f_{i+1}^2 \circ - \end{bmatrix}} \begin{matrix} \text{Hom}(C_i^2, C_i^1) \\ \oplus \\ \text{Hom}(C_i^2, C_i^2) \end{matrix} \xrightarrow{\begin{bmatrix} f_i^1 \circ - & 0 \end{bmatrix}} \text{Hom}(C_i^2, C_{i-1})$$

is exact in the middle, which implies that $f_{i+1}^2 \circ -$ is surjective on $\text{Hom}(C_i^2, C_i^2)$, and so there is $h \in \text{Hom}(C_i^2, C_{i+1})$ such that $f_{i+1}^2 \circ h = \text{id}_{C_i^2}$. Since $f_{i+1}^2 \in \text{rad}(C_{i+1}, C_i)$, it follows that $C_i^2 = 0$ and we are done. For proving the case $i = 1$, just replace $\text{Hom}(C_i^2, C_{i+1})$ with $\text{rad}(C_i^2, C_{i+1})$, and the argument goes through.

The dual argument, using the fact that d -almost split sequences are source, yields the claim for rows. \square

4.3. Tensor products

The main tool which allows us to perform homological computations for tensor products is the Künneth formula over a field ([2, VI.3.3.1]):

Lemma 4.9. *If X_\bullet, Y_\bullet are complexes, then there is a functorial isomorphism*

$$H_i(X_\bullet \otimes Y_\bullet) \cong \bigoplus_{p+q=i} H_p(X_\bullet) \otimes H_q(Y_\bullet).$$

Since tensor products of projective resolutions are projective resolutions, we immediately get

Lemma 4.10. *If $M_1, M_2 \in \text{mod } A$ and $N_1, N_2 \in \text{mod } B$, then there is a functorial isomorphism*

$$\text{Ext}_{A \otimes B}^i(M_1 \otimes N_1, M_2 \otimes N_2) \cong \bigoplus_{p+q=i} \text{Ext}_A^p(M_1, M_2) \otimes \text{Ext}_B^q(N_1, N_2).$$

The total tensor product of complexes is a functor in a natural way, so we can speak of tensor products of maps of complexes (for a very general treatment of how this is done, see [2, IV.4 and IV.5]). An important result which is proved in [16] for d -representation finite algebras is also true for d -complete algebras, namely:

Theorem 4.11. *Let A, B be n - respectively m -complete algebras. Let $\text{Cone } \varphi$ and $\text{Cone } \psi$ be n - respectively m -almost split sequences starting in $\text{add } \tau_n^i DA$ respectively $\text{add } \tau_m^i DB$ for some common $i > 0$. Then $\text{Cone}(\varphi \otimes \psi)$ is an $(n + m)$ -almost split sequence in $\mathcal{M}(A \otimes B)$.*

Proof. This is proved in the same way as in [16, Section 3.3]. For convenience, we present the main points of the proof. By definition $\text{Cone}(\varphi \otimes \psi)$ is a complex bounded between 0 and $n + m + 1$, it is exact by the Künneth formula, and it is easy to check that all maps appearing are radical. Now $\varphi : A_\bullet^0 \rightarrow A_\bullet^1$ and $\psi : B_\bullet^0 \rightarrow B_\bullet^1$, and by assumption we have that $A_j^0 \in \text{add } \tau_n^i DA$, $A_j^1 \in \text{add } \tau_n^{i-1} DA$, $B_j^0 \in \text{add } \tau_m^i DB$ and

$B_j^1 \in \text{add } \tau_m^{i-1}DB$ for every j since $A_j \otimes B_j \in \mathcal{M}(A \otimes B)$. Let now $M \otimes N$ be any indecomposable in $\mathcal{M}(A \otimes B)$. We need to prove that $F_{M \otimes N}(\text{Cone}(\varphi \otimes \psi))$ is exact. As in [16, Section 2.3], for a radical map of radical complexes $\eta : A_\bullet \rightarrow B_\bullet$ and a module X we can define $\tilde{F}_X(\eta) = \eta \circ - : \text{Hom}(X, A_\bullet) \rightarrow F_X(B_\bullet)$. Then in our setting there is a commutative diagram

$$\begin{CD} \text{Hom}(M, A_\bullet^0) \otimes \text{Hom}(N, B_\bullet^0) @>\cong>> \text{Hom}(M \otimes N, A_\bullet^0 \otimes B_\bullet^0) \\ @V{\tilde{F}_M(\varphi) \otimes \tilde{F}_N(\psi)}VV @VV{\tilde{F}_{M \otimes N}(\varphi \otimes \psi)}V \\ F_M(A_\bullet^1) \otimes F_N(B_\bullet^1) @>>> F_{M \otimes N}(A_\bullet^1 \otimes B_\bullet^1). \end{CD}$$

Now $F_{M \otimes N}(\text{Cone}(\varphi \otimes \psi))$ is exact if and only if $\tilde{F}_{M \otimes N}(\varphi \otimes \psi)$ is a quasi-isomorphism. The left map in the diagram $\tilde{F}_M(\varphi) \otimes \tilde{F}_N(\psi)$ is a quasi-isomorphism since $\text{Cone}(\varphi)$ and $\text{Cone}(\psi)$ are n - respectively m -almost split sequences. Then it is enough to prove that the bottom map is a quasi-isomorphism, and this is done by showing that its cokernel is isomorphic to

$$F_M(A_\bullet^1) \otimes \text{top}(N, B_0^1) \oplus \text{top}(M, A_0^1) \otimes F_N(B_\bullet^1)$$

and then by easy verification that the above cokernel is exact. The computation of the cokernel is done explicitly in [16, Section 3.3, pp. 660–662]. \square

Corollary 4.12. *Let A, B be n - respectively m -complete algebras. Then $\mathcal{M}(A \otimes B)$ has $(n + m)$ -almost split sequences.*

Notice that the above theorem does not require the algebra $A \otimes B$ to be $(n + m)$ -representation finite (in which case we know a priori that $(n + m)$ -almost split sequences must exist). In the setting of [16], this result is about describing the structure of such sequences. In the setting of d -complete algebras, this result is used to prove that $(n + m)$ -almost split sequences exist, whereas it is a priori not clear that they should.

One can also say something about injective modules (which are not the starting point of any d -almost split sequence).

Proposition 4.13. *Let A, B be n - respectively m -complete algebras, and let $\Lambda = A \otimes B$. Then for every injective Λ -module $X \otimes Y$ there is a source sequence*

$$X \otimes Y \rightarrow E_{n+m} \rightarrow \cdots \rightarrow E_1 \rightarrow 0$$

in $\mathcal{M}(\Lambda)$.

Proof. Since X and Y are injective, we have sequences in $\mathcal{M}(A)$ respectively $\mathcal{M}(B)$

$$\begin{aligned} X_\bullet &= X \rightarrow C_n \rightarrow \cdots \rightarrow C_1 \rightarrow 0 \\ Y_\bullet &= Y \rightarrow D_m \rightarrow \cdots \rightarrow D_1 \rightarrow 0 \end{aligned}$$

such that

$$\begin{aligned} 0 \rightarrow \text{Hom}(C_1, M) \rightarrow \cdots \rightarrow \text{Hom}(X, M) \rightarrow \text{top}(X, M) \rightarrow 0, \\ 0 \rightarrow \text{Hom}(D_1, N) \rightarrow \cdots \rightarrow \text{Hom}(Y, N) \rightarrow \text{top}(Y, N) \rightarrow 0 \end{aligned}$$

are exact for all indecomposables M, N . Now consider the homology of $X_\bullet \otimes Y_\bullet$.

$$H_i(X_\bullet \otimes Y_\bullet) = \bigoplus_{p+q=i} H_p(X_\bullet) \otimes H_q(Y_\bullet) = \begin{cases} H_0(X_\bullet) \otimes H_0(Y_\bullet) & \text{if } i = n + m + 2 \\ 0 & \text{else.} \end{cases}$$

So we have at least an exact sequence

$$X_\bullet \otimes Y_\bullet = X \otimes Y \rightarrow \cdots \rightarrow C_1 \otimes D_1 \rightarrow 0.$$

Apply $\text{Hom}(-, M \otimes N)$ to this sequence and compute homology.

$$\begin{aligned} H_i(\text{Hom}(X_\bullet \otimes Y_\bullet, M \otimes N)) &= H_i(\text{Hom}(X_\bullet, M) \otimes \text{Hom}(Y_\bullet, M)) = \\ &= \bigoplus_{p+q=i} H_p(\text{Hom}(X_\bullet, M)) \otimes H_q(\text{Hom}(Y_\bullet, M)) = \\ &= \begin{cases} \text{top}(X, M) \otimes \text{top}(Y, N) & \text{if } i = 0 \\ 0 & \text{else.} \end{cases} \end{aligned}$$

We will be done if we prove that $X_\bullet \otimes Y_\bullet$ is source, which amounts now to prove that

$$\text{top}(X \otimes Y, M \otimes N) = H_0(\text{Hom}(X_\bullet \otimes Y_\bullet, M \otimes N)) = \text{top}(X, M) \otimes \text{top}(Y, N).$$

By tensoring the complexes

$$0 \rightarrow \text{rad}(X, M) \rightarrow \text{Hom}(X, M)$$

and

$$0 \rightarrow \text{rad}(Y, N) \rightarrow \text{Hom}(Y, N)$$

and looking at homology, one finds an exact sequence

$$\begin{aligned} 0 \rightarrow \text{rad}(X, M) \otimes \text{Hom}(Y, N) + \text{Hom}(X, M) \otimes \text{rad}(Y, N) \rightarrow \\ \rightarrow \text{Hom}(X, M) \otimes \text{Hom}(Y, N) \rightarrow \text{top}(X, M) \otimes \text{top}(Y, N) \rightarrow 0. \end{aligned}$$

Now the middle term is isomorphic to $\text{Hom}(X \otimes Y, M \otimes N)$, and this isomorphism induces an isomorphism between the first term and $\text{rad}(X \otimes Y, M \otimes N)$, hence by looking at cokernels we get

$$\begin{aligned} \text{top}(X \otimes Y, M \otimes N) &\cong \frac{\text{Hom}(X \otimes Y, M \otimes N)}{\text{rad}(X \otimes Y, M \otimes N)} \\ &\cong \frac{\text{Hom}(X, M) \otimes \text{Hom}(Y, N)}{\text{rad}(X, M) \otimes \text{Hom}(Y, N) + \text{Hom}(X, M) \otimes \text{rad}(Y, N)} \\ &\cong \text{top}(X, M) \otimes \text{top}(Y, N) \end{aligned}$$

and we are done. \square

Lemma 4.14. *Let A, B be n - respectively m -complete algebras. Then the following are equivalent:*

(1) $T_{A \otimes B} \cong T_A \otimes T_B$.

(2) $\exists l \in \mathbb{N}$ such that A and B are l -homogeneous.

Proof. (2) \Rightarrow (1) is clear by definition.

To prove (1) \Rightarrow (2), assume it does not hold, that is $T_{A \otimes B} \cong T_A \otimes T_B$ but there are i, j such that $l_i \neq l_j$ for the corresponding indecomposable injectives $E_i \in \text{add } DA$ and $F_j \in \text{add } DB$. We can assume that $l_i > l_j$, otherwise the proof is similar. Call $X_{ij} = \tau_n^{l_i-1} E_i \otimes \tau_m^{l_j-1} F_j \in \text{add } T_{A \otimes B}$. Then

$$\tau_{m+n}^{-l_j+1}(X_{ij}) = \tau_n^{l_i-l_j} E_i \otimes F_j$$

is not injective, since by assumption $\tau_n^{l_i-l_j} E_i$ is not injective. On the other hand, modules in $\mathcal{M}(A \otimes B)$ which satisfy $\tau_{m+n} X = 0$ are precisely the injective $A \otimes B$ -modules, and so $\tau_{m+n}^{-l_j+1}(X_{ij})$ is not in \mathcal{M} , contradiction. \square

4.4. Acyclicity

We collect here some lemmas about acyclicity which we will use.

Lemma 4.15. *The module Λ_Λ is directed if and only if the module $D_\Lambda \Lambda$ is directed.*

Proof. The Nakayama functor induces an equivalence $\nu : \text{add } \Lambda_\Lambda \rightarrow \text{add } D_\Lambda \Lambda$, and the definition of directedness is invariant under equivalence. \square

Lemma 4.16. *Let Λ be d -complete. Then Λ is acyclic if and only if \mathcal{M} is directed.*

Proof. If \mathcal{M} is directed, then so is $\text{add } D\Lambda \subseteq \mathcal{M}$. By Lemma 4.15, Λ is then acyclic.

Conversely, if Λ is acyclic then $\text{add } D\Lambda$ is directed by Lemma 4.15, and then so is $\text{add } \tau_d^i D\Lambda$ for any i by Lemma 4.3. Any nonzero map between indecomposables in \mathcal{M} is either within a slice $\mathcal{S}(i) = \text{add } \tau_d^i D\Lambda$ or from $\mathcal{S}(i)$ to $\mathcal{S}(j)$ with $j < i$. Therefore there can be no cycles within a slice nor cycles that contain modules from different slices and \mathcal{M} is directed. \square

Acyclicity is well suited to study d -almost split sequences.

Lemma 4.17. *Let Λ be d -complete, and let*

$$0 \longrightarrow \tau_d X \longrightarrow C_d \longrightarrow \cdots \longrightarrow C_1 \longrightarrow X \longrightarrow 0$$

be a d -almost split sequence in $\text{mod } \Lambda$. Then for every indecomposable summand Y of $\bigoplus_{i=1}^d C_i$, we have

$$\tau_d X < Y < X.$$

Proof. This follows directly from Lemma 4.8 and the definition of $<$. \square

Let us now consider acyclicity in relation to tensor products.

Lemma 4.18. *The algebras A and B are acyclic if and only if $\Lambda = A \otimes B$ is acyclic.*

Proof. Let us first remark that for $X, X' \in \text{mod } A$ and $Y, Y' \in \text{mod } B$ we have

$$\text{rad}(X \otimes Y, X' \otimes Y') = \text{rad}(X, X') \otimes \text{Hom}(Y, Y') + \text{Hom}(X, X') \otimes \text{rad}(Y, Y')$$

by [16, Lemma 3.6]. Assume $X < X$ in $\text{add } A$ via X_1, \dots, X_m . Then for an indecomposable $P \in \text{add } B$ we have that $X \otimes P < X \otimes P$ via $X_1 \otimes P, \dots, X_m \otimes P$ since

$$\text{rad}(X_i \otimes P, X_{i+1} \otimes P) \supseteq \text{rad}(X_i, X_{i+1}) \otimes \text{End}(P) \neq 0$$

for all i . Therefore if Λ is acyclic then A is acyclic. By symmetry, if Λ is acyclic then B is acyclic as well.

Let us now prove the converse implication. Assume that $X \otimes Y < X \otimes Y$ in $\text{add } \Lambda$ via $X_1 \otimes Y_1, \dots, X_m \otimes Y_m$. We can assume that $\text{rad}(X, X) = 0 = \text{rad}(Y, Y)$. Moreover, it cannot be that $X_i \cong X$ for all i and that $Y_j \cong Y$ for all j . Without loss of generality, assume that $X_i \not\cong X$ for some i . We will prove that $X < X$ via a subsequence (Z_j) of the X_i 's. We have that $\text{Hom}(X_i, X_{i+1}) \neq 0$ for all i by assumption. Set $Z_0 = X$ and $Z_j = X_i$, where $i = \min \{l \mid X_l \not\cong Z_{j-1}\}$ for $j > 0$. By construction, $Z_p = X$ for some p (and for $j > p$, Z_j is not defined). Then we are done, since by construction $\text{Hom}(Z_i, Z_{i+1}) \neq 0$ and $Z_i \not\cong Z_{i+1}$ so that $\text{rad}(Z_i, Z_{i+1}) \neq 0$ since Z_i, Z_{i+1} are indecomposable. \square

5. Proof of main result

From now on, let A be n -complete acyclic, let B be m -complete acyclic and let $\Lambda = A \otimes_k B$. We use the notation of Definition 2.3. There are three conditions that need to be checked to prove the main theorem (since we saw in Lemma 4.18 that Λ is acyclic), namely that properties $(A_d), (B_d), (C_d)$ in Definition 2.3 are preserved under tensor products.

Proposition 5.1. $\text{Ext}_\Lambda^i(\mathcal{M}, \mathcal{M}) = 0$ for $0 < i < n + m$.

Proof. Let $X \otimes Y \in \mathcal{M}_P$. We have for $i < n + m$

$$\text{Ext}^i(X \otimes Y, A \otimes B) = \bigoplus_{p+q=i} \text{Ext}^p(X, A) \otimes \text{Ext}^q(Y, B) = 0$$

so we conclude by [12, Proposition 2.5 (a)]. \square

By the same formula, Λ satisfies condition (C_{n+m}) :

Lemma 5.2. $\text{Ext}_\Lambda^i(\mathcal{M}_P, \Lambda) = 0$ for all $0 < i < n + m$.

Proof. Use the same formula as in Proposition 5.1. \square

Notice that since $\tau_{n+m} = \tau_n \otimes \tau_m$ on \mathcal{M} , for sufficiently big l we have $\tau_{n+m}^l D\Lambda = 0$, so \mathcal{M} has an additive generator.

We now start proving that condition (A_{n+m}) holds.

For $S = S_1 \oplus S_2$ with $S_1 \in \text{add } T$ and $S_2 \in \mathcal{M}_P$, define $ES = S_1 \oplus \tau_{n+m} S_2$. Note that $E^l D\Lambda = T$ for $l \gg 0$. Now fix $S = E^i D\Lambda$ for some $i \geq 0$. To check condition (A_{n+m}) for Λ , we need some preliminaries.

Lemma 5.3. If $\text{Ext}_\Lambda^i(S, S) = 0$ for all $i \neq 0$, then $\text{Ext}_\Lambda^i(ES, ES) = 0$ for all $i \neq 0$.

Proof. Since $\text{Ext}_\Lambda^i(\mathcal{M}, \mathcal{M}) = 0$ for $0 < i < n + m$, it suffices to check that $\text{Ext}^{n+m}(ES, ES) = 0$. Since $ES = S_1 \oplus \tau_{n+m} S_2$, consider first $M_1 \otimes N_1 \in \text{add } S_1$ and $M_2 \otimes N_2 \in \text{add } ES$. Then

$$\text{Ext}^{n+m}(M_1 \otimes N_1, M_2 \otimes N_2) = \text{Ext}^n(M_1, M_2) \otimes \text{Ext}^m(N_1, N_2) = 0$$

since $M_1 \otimes N_1 \in \text{add } S_1 \subseteq \text{add } T$ implies that either M_1 or N_1 is relative projective in T_A^\perp respectively T_B^\perp . This proves that $\text{Ext}^{n+m}(S_1, ES) = 0$. Now let Y be an indecomposable summand of ES , and consider $\text{Ext}^{n+m}(\tau_{n+m}S_2, Y)$. If Y is injective, then this is 0. Otherwise, $Y = \tau_{n+m}\tau_{n+m}^-Y$ and

$$\text{Ext}^{n+m}(\tau_{n+m}S_2, Y) = \text{Ext}^{n+m}(S_2, \tau_{n+m}^-Y) = 0$$

by the assumption. \square

Lemma 5.4. *If S is tilting then thick $ES = \mathcal{D}^b(\Lambda)$.*

Proof. Set $\mathcal{S} = \text{add } S$. For $X \in \text{ind } \mathcal{S}$, define $h(X)$ to be the height of X with respect to the partial order introduced in Section 4.4 on $\text{ind } \mathcal{S}$ (here it is crucial that Λ be acyclic, which follows from the assumptions on A and B and Lemma 4.18), that is

$$h(X) = \max \{n \mid \exists Y_0 < \dots < Y_n = X, Y_i \in \text{ind } \mathcal{S}\}.$$

Notice that $X > Y$ implies $h(X) > h(Y)$, and the reverse implication holds provided that X and Y are comparable. Call $\mathcal{C}_i = \text{add}(\{ES\} \cup \{Y \in \text{ind } \mathcal{S} \mid h(Y) < i\})$. For $X \in \text{ind } \mathcal{S}$, if $\tau_{n+m}X = 0$ then $X \in \text{add } ES$. Otherwise, there is an $(n + m)$ -almost split sequence

$$0 \rightarrow \tau_{n+m}X \rightarrow \dots \rightarrow X \rightarrow 0$$

whose middle terms are in $\text{add}(\{ES\} \cup \{Y \in \text{ind } \mathcal{S} \mid Y < X\})$ by Lemma 4.17. In particular if $h(X) \leq i$ then the middle terms in the sequence are in

$$\text{add}(\{ES\} \cup \{Y \in \text{ind } \mathcal{S} \mid h(Y) < i\}) = \mathcal{C}_i.$$

It follows that $\text{thick } \mathcal{C}_{i+1} \subseteq \text{thick } \mathcal{C}_i$, so $\text{thick } \mathcal{C}_j \subseteq \text{thick } \mathcal{C}_0$ for every j . Now $\mathcal{C}_0 = \text{add } ES$, and $\mathcal{C}_j = \text{add}(ES \oplus S)$ for some j , so we get that $\text{thick } ES = \text{thick } \mathcal{C}_0 = \text{thick } \mathcal{C}_j = \mathcal{D}^b(\Lambda)$ as claimed. \square

Theorem 5.5. *$T = T_{A \otimes B}$ is tilting.*

Proof. By Lemma 5.3 and Lemma 5.4, if $S = E^i DA$ is tilting then $ES = E^{i+1} DA$ is tilting. Since DA is tilting, and $T = E^l DA$ for some l , it follows that T is tilting. \square

Now we start proving that condition (B_{n+m}) holds. We will use the following result (this is [12, Theorem 2.2(b)]):

Theorem 5.6. *Let Λ be a finite-dimensional k -algebra, $d \geq 1$ and $T \in \text{mod } \Lambda$ a tilting module with $\text{proj. dim } T \leq d$. Let $\mathcal{C} = \text{add } C$ be a subcategory of T^\perp such that $\text{Ext}_\Lambda^i(\mathcal{C}, \mathcal{C}) = 0$ for $0 < i < d$ and $T \oplus DA \in \mathcal{C}$. Then the following are equivalent:*

- (1) \mathcal{C} is a d -cluster tilting subcategory in T^\perp .
- (2) Every indecomposable $X \in \mathcal{C}$ has a source sequence of the form

$$X \rightarrow C_d \rightarrow \dots \rightarrow C_0 \rightarrow 0$$

with $C_i \in \mathcal{C}$ for all i .

We want to apply this to $\Lambda = A \otimes B$, $\mathcal{C} = \mathcal{M}$, $T = T_{A \otimes B}$ and $d = n + m$.

Lemma 5.7. $\mathcal{M} \subseteq T^\perp$.

Proof. By Proposition 5.1, it is enough to check that $\text{Ext}^{n+m}(T, \mathcal{M}) = 0$. Let $M_1 \otimes M_2 \in \text{add } T$. Then either M_1 or M_2 is relative projective in T_A^\perp respectively T_B^\perp , so

$$\text{Ext}^{n+m}(M_1 \otimes M_2, N_1 \otimes N_2) = \text{Ext}^n(M_1, N_1) \otimes \text{Ext}^m(M_2, N_2) = 0$$

for any $N_1 \otimes N_2 \in \mathcal{M}$. \square

Theorem 5.8. \mathcal{M} is an $(n + m)$ -cluster tilting subcategory of T^\perp .

Proof. By Proposition 5.1 and Lemma 5.7, we can take $\Lambda = A \otimes B$, $\mathcal{C} = \mathcal{M}$, $T = T_{A \otimes B}$ and $d = n + m$ in the assumptions of Theorem 5.6. By Corollary 4.12 and Proposition 4.13, condition (2) is satisfied. Our claim is then the equivalent statement (1). \square

Now we have established everything we need to prove the main result.

Proof of Theorem 3.1. By Theorem 5.5, Theorem 5.8, and Lemma 5.2, we have that $A \otimes B$ satisfies the conditions $(A_{n+m}), (B_{n+m}), (C_{n+m})$ in the definition of $(n + m)$ -complete algebra. By Lemma 4.18, $A \otimes B$ is acyclic. \square

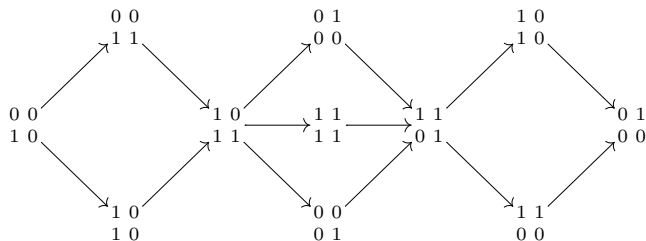
Proof of Corollary 3.2. By Theorem 3.1, $A \otimes B$ is $(n + m)$ -complete. By [12, Proposition 1.13], we have that $T_A \cong A$, $T_B \cong B$ and that $A \otimes B$ is $(n + m)$ -representation finite if and only if $T_{A \otimes B} \cong A \otimes B$. By Lemma 4.14, this happens if and only if A and B are l -homogeneous for some common l . \square

6. Examples

Let us consider one of the simplest non-trivial examples. Let $A = B = kQ$, where Q is the quiver

$$1 \longleftarrow 2.$$

Then $\Lambda = A \otimes B$ is the quiver algebra of a commutative square. This algebra is 2-complete, since the factors are 1-representation finite. It is not 2-representation finite since the factors are not homogeneous. However, Λ is representation finite, so we can draw the entire Auslander–Reiten quiver of Λ . We represent modules by their dimension vector.



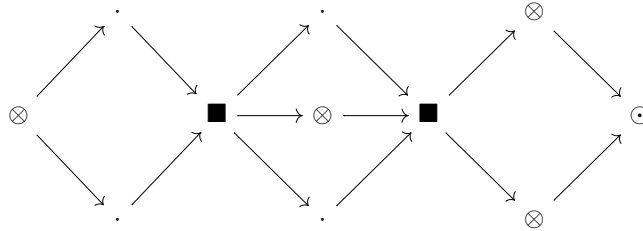
In this case,

$$T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

and

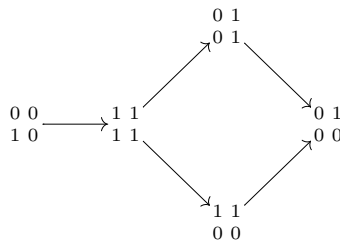
$$\mathcal{M} = \text{add } M = \text{add}(T \oplus \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}).$$

One can explicitly compute all Ext-groups of all pairs of indecomposables, since we have only finitely many. If we represent by \otimes the indecomposables in $\text{add } T$, by \odot the ones in \mathcal{M} but not in $\text{add } T$, by \blacksquare the ones in T^\perp but not in \mathcal{M} , and by \cdot the ones outside T^\perp , we get the following picture:



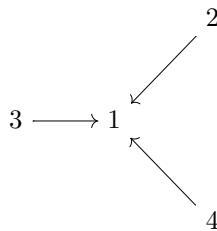
It can be checked that both the indecomposable modules in $T^\perp \setminus \mathcal{M}$ have extensions with M on both sides, as it is required by the definition of 2-cluster tilting. Here we find that \mathcal{M} is 2-cluster tilting in T^\perp .

The Auslander–Reiten quiver of $\text{add}(M)$ is given by

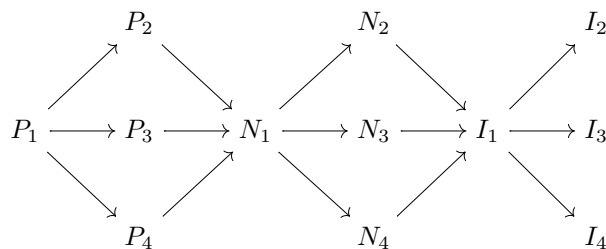


and this is also a picture of the only 2-almost split sequence we have.

As a second example, consider the quiver Q' :



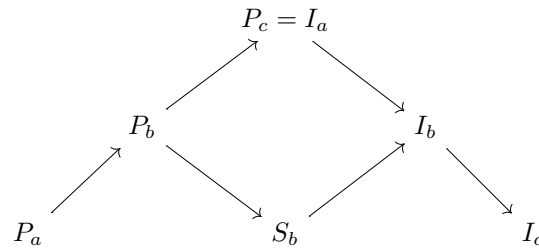
and the corresponding path algebra $A' = kQ'$. The Auslander–Reiten quiver of A' looks like



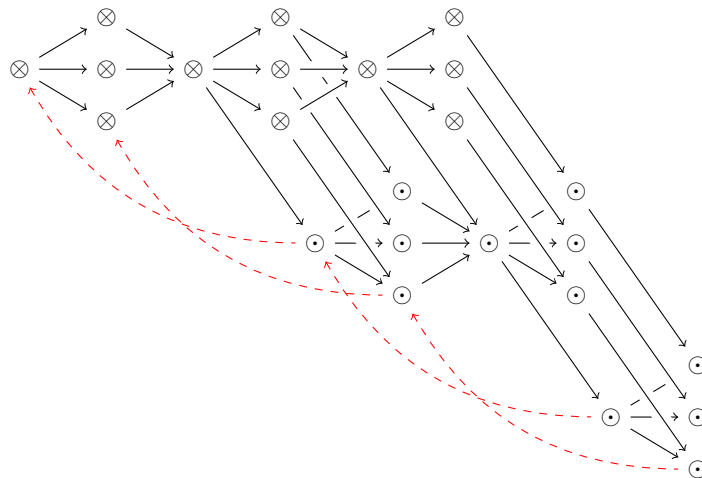
We take $B' = kQ''$, where Q'' is the quiver

$$a \longleftarrow b \longleftarrow c$$

The Auslander–Reiten quiver of B' looks like



These algebras are both 1-representation finite, so in particular they are 1-complete. Their tensor product $\Lambda' = A' \otimes B'$ is therefore 2-complete. It is not 2-representation finite since B' is not homogeneous. In this example, we cannot draw the entire module category of Λ' , but we still have complete control over the “higher Auslander–Reiten quiver” of Λ' , that is the Auslander–Reiten quiver of $\text{add}(M)$:



Here the dashed arrows represent τ_2 , and we have drawn them only between some modules to avoid clogging the picture. We have again written \otimes for indecomposable summands of T , and \odot for the other indecomposable summands of M . It should be clear from the picture which module corresponds to which node.

Notice that this example presents some regularity which is not to be expected in general, since we have taken A' to be homogeneous. Moreover, in this example (and in general) we cannot directly check that arbitrary modules in $\text{mod } \Lambda'$ which are in T^\perp have extensions on both sides with \mathcal{M} .

Acknowledgements

I am especially thankful to his advisor Martin Herschend for the many helpful comments and suggestions. I would also like to thank an anonymous referee for his or her comments.

References

[1] Claire Amiot, Osamu Iyama, Idun Reiten, Stable categories of Cohen–Macaulay modules and cluster categories, *Am. J. Math.* 137 (3) (2015) 813–857.
 [2] Henri Cartan, Samuel Eilenberg, *Homological Algebra*, Princeton University Press, Princeton, NJ, 1956.
 [3] Christof Geiss, Bernhard Keller, Steffen Oppermann, n -angulated categories, *J. Reine Angew. Math.* 675 (2013) 101–120.
 [4] Martin Herschend, Osamu Iyama, n -representation-finite algebras and twisted fractionally Calabi–Yau algebras, *Bull. Lond. Math. Soc.* 43 (3) (2011) 449–466.

- [5] Martin Herschend, Osamu Iyama, Selfinjective quivers with potential and 2-representation-finite algebras, *Compos. Math.* 147 (6) (2011) 1885–1920.
- [6] Martin Herschend, Osamu Iyama, Hiroyuki Minamoto, Steffen Oppermann, Representation theory of Geigle–Lenzing complete intersections, arXiv:1409.0668v1, 2014.
- [7] Martin Herschend, Osamu Iyama, Steffen Oppermann, n -representation infinite algebras, *Adv. Math.* 252 (2014) 292–342.
- [8] Osamu Iyama, Steffen Oppermann, n -representation-finite algebras and n -APR tilting, *Trans. Am. Math. Soc.* 363 (12) (2011) 6575–6614.
- [9] Osamu Iyama, Auslander correspondence, *Adv. Math.* 210 (1) (2007) 51–82.
- [10] Osamu Iyama, Higher-dimensional Auslander–Reiten theory on maximal orthogonal subcategories, *Adv. Math.* 210 (1) (2007) 22–50.
- [11] Osamu Iyama, Auslander–Reiten theory revisited, in: *Trends in Representation Theory of Algebras and Related Topics*, in: EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2008, pp. 349–397.
- [12] Osamu Iyama, Cluster tilting for higher Auslander algebras, *Adv. Math.* 226 (1) (2011) 1–61.
- [13] Gustavo Jasso, n -Abelian and n -exact categories, *Math. Z.* 283 (3–4) (2016) 703–759.
- [14] Gustavo Jasso, Sondre Kvamme, An introduction to higher Auslander–Reiten theory, arXiv:1610.05458v1, 2016.
- [15] Peter Jørgensen, Torsion classes and t-structures in higher homological algebra, arXiv:1412.0214v3, 2015.
- [16] Andrea Pasquali, Tensor products of higher almost split sequences, *J. Pure Appl. Algebra* 221 (3) (2017) 645–665.