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Constructions in higher-dimensional Auslander-Reiten theory

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Abstract

Pasquali, A. 2019. Constructions in higher-dimensional Auslander-Reiten theory. *Uppsala Dissertations in Mathematics* 114. 42 pp. Uppsala: Acta Universitatis Upsaliensis. ISBN 978-91-506-2754-1.

This thesis consists of an introduction and five research articles about representation theory of algebras.

Papers I and II focus on the tensor product of algebras from the point of view of higher-dimensional Auslander-Reiten theory. In Paper I we consider the tensor product Λ of two algebras which are n- respectively m-representation finite. In the case when Λ itself is $(n+m)$-representation finite, we construct its $(n+m)$-almost split sequences explicitly in function of the n- and m-almost split sequences of the factors. In Paper II we use the constructions of Paper I to prove the following result: the tensor product of an n- and an m-complete acyclic algebras (in the sense of Iyama) is $(n+m)$-complete and acyclic.

Papers III and IV deal with the combinatorics of Postnikov diagrams, or equivalently of the Grassmannian cluster category. This is motivated by 2-dimensional Auslander-Reiten theory: we are interested in constructing self-injective Jacobian algebras as they are the 3-preprojective algebras of 2-representation finite algebras. In Paper III we investigate when the stable Jacobian algebra associated to a (k,n)-Postnikov diagram is self-injective. We prove that this happens if and only if the Postnikov diagram is invariant under rotation by $2\pi k/n$. In Paper IV (joint with Thörnblad and Zimmermann) we determine a necessary and sufficient condition on (k,n) for such a symmetric Postnikov diagram to exist, namely $k \equiv -1$, 0 or 1 modulo $n/GCD(k,n)$. As a corollary, we prove that there exist self-injective planar quivers with potential with Nakayama automorphism of any prescribed order, answering a question by Herschend and Iyama.

Paper V (joint with Giovannini) is about skew group algebras. Let G be a finite group acting on a quiver with potential (Q, W), such that certain assumptions hold. We construct a quiver with potential (Q_G, W_G) such that the skew group algebra of the Jacobian algebra of (Q, W) is Morita equivalent to the Jacobian algebra of (Q_G, W_G). Moreover, we show that this construction is a duality if G is abelian. We also apply our results to quivers with potential associated to Postnikov diagrams.

Keywords: Representation theory, higher-dimensional Auslander-Reiten theory, Postnikov diagram, 2-representation finite algebra, self-injective algebra, quiver with potential, skew group algebra

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né dolcezza di figlio, né la pieta
del vecchio padre, né ’l debito amore
lo qual dovea Penelope far lieta,
vincer potero dentro a me l’ardore
ch’i’ ebbi a divenir del mondo esperto
e de li vizi umani e del valore;
ma misi me per l’alto mare aperto
sol con un legno e con quella compagna
picciola da la qual non fui diserto.

—Dante, Inferno XXVI

List of papers

This thesis is based on the following papers, which are referred to in the text by their roman numerals.

I A. Pasquali, Tensor products of higher almost split sequences. *J. Pure Appl. Algebra* **221** (2017), 645–665.

II A. Pasquali, Tensor products of n-complete algebras. *J. Pure Appl. Algebra* **223** (2019), 3537–3553.

III A. Pasquali, Self-injective Jacobian algebras from Postnikov diagrams. *Algebr. Represent. Theory* (2019), https://doi.org/10.1007/s10468-019-09882-8.

IV A. Pasquali, E. Thörnblad, and J. Zimmermann. Existence of symmetric maximal noncrossing collections of k-element subsets. arXiv:1808.03556, submitted for publication.

V S. Giovannini and A. Pasquali. Skew group algebras of Jacobian algebras. *J. Algebra* **526** (2019), 112–165.

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Contents

1. Foreword

Fatti non foste a viver come bruti, ma per seguir virtute e canoscenza.

—Dante, Inferno XXVI

Mathematics is not an empirical science. There are hypotheses and predictions and educated guesses, but there are no experiments. The real world is not a concern of mathematical research, which instead focuses on the world of ideas. The analogue of an experiment in this world is indeed testing a theory against an example, but this takes place in one's mind (and partially on coffee-stained paper) and not in a laboratory.

The process of designing an experiment in the sciences requires a deep understanding of the theory, so that one can exclude all non-relevant factors and expect one result if the theory is correct and another result if the theory is wrong. There are limitations in the form of available resources and technology.

The process of choosing examples in mathematics is similar. They should be simple enough that computations are possible, yet deep and structured enough that one expects them to either falsify a hypothesis or confirm it in a non-trivial way. There are limitations in the form of computabiliy, checkability of properties, and availability of examples in the first place.

How are examples created (or found, depending on one's personal Weltanschauung)? One can often try to construct "the" prototypical example of a certain object or phenomenon, by finding something that has the properties in question "and no more". One can also, often, look for examples in adjacent areas of research, in hope to shed light not only on the topic but also on hidden connections that might be of interest themselves.

The latter is what this thesis is in essence about. It is a collection of theorems and constructions which in the end leaves us with more examples than we had before. The original motivation of all the papers comprised in this thesis comes from higher-dimensional Auslander-Reiten theory, a very specific subspecialty of representation theory of algebras. However, the constructions involved are freely borrowed from other areas of algebra (broadly interpreted). On the one hand, this allows for novelty, finding expected structure in unexpected places. On the other hand, maybe this structure is there for a reason. And by understanding these reasons one can hope to reveal hidden links and connections between different subjects in algebra. From my point of view, this thesis is thus a worthy contribution to an exciting and still to a large extent unexplored part of human knowledge.

The structure is as follows. The core of this thesis consists of five research papers that I wrote, with coauthors, during my time as a doctoral student. These articles are attached at the end. The next chapters are devoted to presenting and recalling some of the necessary background, and then stating the main results of the papers. Then the reader will find a short perspective on possible further research directions, as well as a summary in Swedish. Last but not least, in Chapter 6 I express my gratitude to everyone who accompanied me in this journey.

2. Preliminaries

The purpose of this chapter is twofold. First, it contains a condensed list of definitions and results which are necessary in order to state the results of the papers. Second, it provides a short historical and contextual introduction to some areas of representation theory of algebra (specifically, those relevant for this thesis).

2.1 Finite-dimensional algebras

We will recall some definitions and basic results in representation theory of finite-dimensional algebras. We refer to [17] [4] for a more detailed treatment.

2.1.1 Algebras and modules

Informally, an algebra is a ring with a compatible vector space structure, or a vector space with a nice associative bilinear operation. Formally, let k be a field.

Definition 2.1.1. A *k-algebra* is a ring with identity Λ which is a k-vector space, such that

$$\lambda(ab) = (\lambda a)b = a(\lambda b)$$

for all $a, b \in \Lambda$ and $\lambda \in k$.

In this thesis, the field k is somewhat in the background, so we will often just speak of Λ as an *algebra*. Historically, much of the theory was developed for algebras over an arbitrary artinian ring [5]. For reference, we summarise here the additional assumptions on the field we need to make. In Papers I and II, we need k to be perfect. In Papers III and IV, we work over the field $\mathbb{C}$, and in fact we denote by k a natural number. In Paper V, we need no assumptions for the main results, and we assume $k = \mathbb{C}$ for some of the applications.

Almost all of the algebras appearing in this thesis are finite dimensional (as vector spaces), so we assume that $\dim_k \Lambda < \infty$ from now on. We also assume that Λ is connected, i.e. that it cannot be written as a nontrivial product

$\Lambda \cong \Lambda_1 \times \Lambda_2$. An algebra *morphism* is a linear ring morphism. The *opposite algebra* Λ^{op} of Λ is Λ as a vector space, but with product $\cdot_{op}$ given by $a \cdot_{op} b = ba$.

If M is a vector space, the set $\mathrm{End}(M)$ of linear maps from M to M is an algebra with product given by composition. A *left* Λ*-module* is a vector space M with an algebra morphism $\Lambda \to \mathrm{End}(M)$. A *right* Λ*-module* is a vector space M with an algebra morphism $\Lambda^{op} \to \mathrm{End}(M)$. If M is a left Λ-module, we can see elements of Λ as acting from the left as linear maps on M, and we write $a \cdot m$ or am for the element we obtain if we let $a \in \Lambda$ act on $m \in M$. Similarly, we write $m \cdot a$ or ma if M is a right Λ-module. A *left (respectively right)* Λ*-module morphism* is a linear map $\varphi : M \to N$ such that $\varphi(a \cdot m) = a \cdot \varphi(m)$ for all $a \in \Lambda, m \in M$ (respectively, $\varphi(m \cdot a) = \varphi(m) \cdot a$). The vector space of Λ-module morphisms from M to N is denoted $\mathrm{Hom}_\Lambda(M,N)$. It has a subspace

$$\mathrm{rad}_\Lambda(M,N) = \{f \in \mathrm{Hom}_\Lambda(M,N) \mid \mathrm{id}_M - g \circ f \text{ is invertible} \, \forall g \in \mathrm{Hom}_\Lambda(N,M)\}.$$

Modules over an algebra are also called *representations* of the algebra, hence the name “representation theory” for the study of modules over algebras. Here again, we are mostly interested in finite-dimensional modules. We denote by $\mathrm{mod}\,\Lambda$ and $\Lambda\mathrm{mod}$ the categories of right respectively left finite-dimensional Λ-modules. One could say that the goal of representation theory (of finite-dimensional algebras) is to describe these categories. These are abelian categories, so they admit a notion of direct sum, kernels of morphisms and exactness. Moreover, the Krull-Schmidt theorem ensures that any module can be decomposed completely with respect to direct sums, and that the summands are uniquely determined. Modules which cannot be decomposed further are called *indecomposable*. An algebra Λ is called *representation finite* if $\mathrm{mod}\,\Lambda$ has finitely many indecomposable objects up to isomorphism.

We remark that $\mathrm{mod}\,\Lambda$ and $\Lambda^{op}\mathrm{mod}$ are isomorphic categories. There is moreover a duality between $\mathrm{mod}\,\Lambda$ and $\Lambda\mathrm{mod}$ given by $D = \mathrm{Hom}_k(-,k)$. If ϕ is an automorphism of Λ and $M \in \mathrm{mod}\,\Lambda$, we can define a “twisted” module structure M_ϕ on M by $m \cdot_\phi a = m \cdot \phi(a)$.

An algebra Λ is always a module over itself from both sides, with action given by multiplication. It is called *basic* if it does not have isomorphic summands as a module over itself. Non-basic algebras play an important role in Paper V, but for any algebra Λ there always exists a basic algebra Λ_b such that $\mathrm{mod}\,\Lambda$ is equivalent to $\mathrm{mod}\,\Lambda_b$. Two algebras with equivalent module categories are called *Morita equivalent*.

2.1.2 Quivers with relations

An important tool for studying modules over algebras is the language of quivers and quiver representations. This was introduced in [13] [14] to address the problem of classifying representation finite algebras, and has since become standard.

A *quiver* is a finite directed graph (loops and multiple edges are allowed). For a quiver Q, we denote by Q_0 its set of vertices and by Q_1 its set of arrows (i.e. oriented edges). Given a quiver Q, we can define an algebra kQ in the following way. A basis of kQ is the set of oriented paths in Q (where we declare that there is a path e_i of length 0 at every vertex i). In particular, kQ is finite dimensional if and only if Q has no oriented cycles. Multiplication of paths is given by concatenation if possible, and multiplying two non-composable paths yields 0. This multiplication rule is extended by linearity to elements of kQ which are not paths. The resulting algebra kQ is called the *path algebra* of Q. This is defined even for quivers with oriented cycles, but then it is an infinite dimensional algebra. Since Q_0 is finite, kQ has a unit given by the sum of all paths of length 0.

A representation of a quiver Q is the assignment of a vector space V_i to every vertex $i \in Q_0$ and of a linear map $V_i \to V_j$ to every arrow $i \to j$ in Q_1. The category of representations of Q is equivalent to the category of left kQ-modules. However, not all algebras are isomorphic (or even Morita equivalent) to a path algebra, a fact which motivates the next construction.

Any path algebra kQ has a two-sided ideal J generated by arrows. An ideal I of kQ is called *admissible* if there exists $n \geq 2$ such that $J^n \subseteq I \subseteq J^2$. By factoring out ideals of this form (even if kQ is infinite dimensional) we can in fact construct all basic algebras.

Theorem 2.1.2 ([4, Theorem II.3.7])**.** *Let Q be a (connected) quiver and I an admissible ideal of kQ. Then the algebra kQ/I is a (connected) basic finite-dimensional algebra. If the base field k is algebraically closed, then every (connected) basic finite-dimensional algebra is isomorphic to an algebra of this form.*

From the point of view of representation theory, we only care about algebras up to Morita equivalence, so by this result it is enough to look at path algebras quotiented by admissible ideals. Usually, one speaks of *relations* for a (nicely) chosen set of generators of an admissible ideal, so that Q becomes a *quiver with relations*.

2.1.3 Homological algebra

In this section we recall some homological properties and constructions of the category $\operatorname{mod}\Lambda$. We will only mention the ones we are going to need, but the interested reader is referred to [9] for a deeper treatment of homological algebra.

As we mentioned earlier, $\operatorname{mod}\Lambda$ is an abelian category, which allows us to talk about direct sums, kernels, complexes, and exactness. If $M \in \operatorname{mod}\Lambda$, we denote by $\operatorname{add} M$ the full subcategory of $\operatorname{mod}\Lambda$ whose objects are all direct sums of direct summands of M. Recall that Λ is naturally a right Λ-module, so we can define *projective Λ-modules* to be the objects of $\operatorname{add}\Lambda$. Since Λ is also a left Λ-module, we can dually define *injective Λ-modules* to be the objects of $\operatorname{add} D\Lambda$. If Λ itself is an injective Λ-module, then Λ is called a *self-injective algebra*. Such algebras play an important role in Papers III and V. If Λ is basic and self-injective, then there always exists an automorphism ϕ of Λ and a map $\Lambda \to D\Lambda$ which is simultaneously an isomorphism $\Lambda \cong D\Lambda$ of left Λ-modules and an isomorphism $\Lambda_\phi \cong D\Lambda$ of right Λ-modules. This automorphism is called a *Nakayama automorphism* and is unique as an outer automorphism of Λ.

We need to introduce projective resolutions, see [9, Chapter V]. If $M \in \operatorname{mod}\Lambda$, a *projective resolution* of M is an exact complex

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

such that P_i is projective for all $i \geq 0$. This is not uniquely determined, but there always exists one. The minimal length (in $\mathbb{Z}_{\geq 0} \cup \{\infty\}$) of a projective resolution is an invariant of M called its *projective dimension* and denoted $\operatorname{proj.dim} M$. The supremum of all projective dimensions of all finite-dimensional Λ-modules is an invariant of Λ, called its *global dimension* and denoted $\operatorname{gl.dim}\Lambda$. So $\operatorname{gl.dim}\Lambda = 0$ means that all Λ-modules are projective, i.e. Λ is Morita equivalent to the path algebra of a quiver with no arrows (if $k = \bar{k}$). Similarly, if $k = \bar{k}$, $\operatorname{gl.dim}\Lambda \leq 1$ means that Λ is Morita equivalent to the path algebra of a quiver (necessarily a quiver with no oriented cycles).

If $N \in \operatorname{mod}\Lambda$, we can apply the functor $\operatorname{Hom}_\Lambda(-,N)$ to a projective resolution of M, to get a complex of the form

$$0 \longrightarrow \operatorname{Hom}_\Lambda(P_0,N) \longrightarrow \operatorname{Hom}_\Lambda(P_1,N) \longrightarrow \operatorname{Hom}_\Lambda(P_2,N) \longrightarrow \cdots$$

The cohomology of this complex in position i does not depend on the choice of projective resolution, and it is denoted by $\operatorname{Ext}^i_\Lambda(M,N)$. This construction is in fact functorial in both M and N.

Certain quotient categories of $\operatorname{mod}\Lambda$ will play an important role. The construction we will now explain works for any exact category, a fact which is used in Paper III. Given an exact category $\mathscr{E}$, we can consider the ideal $\mathscr{I}$ consisting of all morphisms factoring through a projective object. The *stable category* $\underline{\mathscr{E}}$ of $\mathscr{E}$ is the quotient category $\mathscr{E}/\mathscr{I}$. Its objects are the same as those of $\mathscr{E}$, and its morphism spaces $\underline{\operatorname{Hom}}(X,Y)$ are the quotients of those of $\mathscr{E}$ by the subspaces of maps $X \to Y$ that factor through a projective object. Dually one can define the *costable category* $\overline{\mathscr{E}}$ of $\mathscr{E}$, by quotienting out morphisms that factor through an injective object.

2.2 Higher-dimensional Auslander-Reiten theory

We will briefly recall some of the rich theory developed by Auslander and Reiten in the end of the 20th century (most of it is collected in [5]). This was then reframed as the "$d = 1$ case" of a more general theory by the school of Iyama [23] [22] [24] [26] [27]. One can interpret the parameter d both as the global dimension of the algebras involved, and in some cases as the dimension of a space in which some object naturally lives. For a suggestive example, see [25].

This higher-dimensional Auslander-Reiten theory has been found to have strong connections to higher homological algebra [28] [15] [30], and has found applications outside representation theory in algebraic geometry [2] [21] [20].

One of the issues of the theory has from the beginning been the scarcity of examples. An important motivation for all the papers included in this thesis was to look for, construct, and study new examples coming from various constructions.

2.2.1 Almost split sequences

In representation theory of finite-dimensional algebras, one of the most important theorems is the existence of almost split sequences. We will now explain the definitions needed to state this if $\mathrm{gl.dim}\,\Lambda \leq 1$, but we remark that the results are true for any global dimension. We choose not to work in the full generality in order to better show where the definitions in dimension d come from. The interested reader is advised to consult [4] and [5] for a much deeper and broader treatment.

Let Λ be an algebra of global dimension at most one. Then for every $X \in \operatorname{mod}\Lambda$, the space $\operatorname{Ext}^1_\Lambda(X,\Lambda)$ is a (left) Λ-module in a natural way. This in fact makes $\tau = D\operatorname{Ext}^1_\Lambda(-,\Lambda)$ into a functor on $\operatorname{mod}\Lambda$. Similarly, $\tau^- = \operatorname{Ext}^1_{\Lambda^{op}}(D-,\Lambda)$ is also a functor on $\operatorname{mod}\Lambda$. These functors are called *Auslander-Reiten translations* and will play a crucial role in describing $\operatorname{mod}\Lambda$. In particular, they appear in the so-called almost split sequences:

Theorem 2.2.1 ([4, Theorem IV.3.1])**.** *Let $L \in \operatorname{mod}\Lambda$ be indecomposable and non-injective. Then there exists a unique (up to isomorphism) exact sequence*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

such that N is indecomposable and the induced sequence

$$0 \longrightarrow \mathrm{Hom}_\Lambda(-,L) \longrightarrow \mathrm{Hom}_\Lambda(-,M) \longrightarrow \mathrm{rad}_\Lambda(-,N) \longrightarrow 0$$

is exact on $\mathrm{mod}\,\Lambda$. *Moreover, in this case $N \cong \tau^- L$ and $L \cong \tau N$. Such a sequence is also determined by the choice of a non-projective indecomposable N, and the induced sequence*

$$0 \longrightarrow \mathrm{Hom}_\Lambda(N,-) \longrightarrow \mathrm{Hom}_\Lambda(M,-) \longrightarrow \mathrm{rad}_\Lambda(L,-) \longrightarrow 0$$

is also exact on $\mathrm{mod}\,\Lambda$.

Such sequences are called *almost split sequences*. The name comes from the following observation: if we replaced rad with Hom in the definition, the condition that $\mathrm{Hom}_\Lambda(-,M) \to \mathrm{Hom}_\Lambda(-,N)$ is surjective would imply that the sequence is split. In fact, $\mathrm{rad}_\Lambda(-,N)$ is the largest possible subspace of $\mathrm{Hom}_\Lambda(-,N)$ such that we can have surjectivity onto it but the sequence does not split. We remark that from the definition it follows that the maps $L \to M$ and $M \to N$ are radical.

If Λ has global dimension (at most) one, this allows us to recursively construct the module category of Λ by starting with the projective indecomposables and constructing almost split sequences (this process is often referred to as "knitting"). In particular, if Λ is representation finite, we can obtain essentially full information about $\mathrm{mod}\,\Lambda$ in this way.

Theorem 2.2.2. *Let Λ be representation finite, with* $\mathrm{gl.dim}\,\Lambda \leq 1$. *Let $P_1,\dots,P_n$ and $I_1,\dots,I_n$ be non-isomorphic representatives of the indecomposable projective respectively injective Λ-modules. Then:*

1. *There is a permutation σ and positive integers $l_1,\dots,l_n$ such that $P_j \cong \tau^{l_j-1} I_{\sigma(j)}$ for all j.*
2. *We have*
$$\mathrm{mod}\,\Lambda = \mathrm{add}\left(\bigoplus_{j=1}^{n}\bigoplus_{p=0}^{l_j-1} \tau^p I_j\right) = \mathrm{add}\left(\bigoplus_{j=1}^{n}\bigoplus_{p=0}^{l_j-1} \tau^{-p} P_j\right).$$
3. *The Auslander-Reiten translations induce quasi-inverse equivalences*
$$\underline{\mathrm{mod}}\Lambda \underset{\tau}{\overset{\tau^-}{\leftrightarrows}} \overline{\mathrm{mod}}\Lambda.$$

2.2.2 d-cluster tilting

In recent years, Iyama and collaborators have developed a version of Auslander-Reiten theory for algebras of higher global dimension. In this setting one looks at a suitable subcategory of $\operatorname{mod}\Lambda$ which has similar homological properties to the whole module category in the classical case, but with all the $\operatorname{Ext}^1_\Lambda$ replaced by $\operatorname{Ext}^d_\Lambda$, where $d = \operatorname{gl.dim}\Lambda$. This subcategory is called a d-cluster tilting subcategory, and if it is generated by a single module then it has finitely many indecomposables up to isomorphism, which means that we can hope to describe it completely using analogous techniques as in the dimension one case.

Precisely, a d-cluster tilting Λ-module is a module M such that

$$\begin{aligned}\operatorname{add} M &= \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(X,M) = 0\,\forall i = 1,\dots,d-1\} = \\ &= \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(M,X) = 0\,\forall i = 1,\dots,d-1\}.\end{aligned}$$

An algebra Λ is called *d-representation finite* if $\operatorname{gl.dim}\Lambda \leq d$ and there exists a d-cluster tilting Λ-module. The category $\operatorname{add} M$ is called a *d-cluster tilting subcategory* of $\operatorname{mod}\Lambda$.

We remark that a 1-cluster tilting module M is such that $\operatorname{add} M = \operatorname{mod}\Lambda$. In particular, Λ is 1-representation finite if and only if Λ is representation finite and $\operatorname{gl.dim}\Lambda \leq 1$. A 2-cluster tilting module is usually just called a cluster tilting module (and in fact this is the origin of the name "d-cluster tilting"). Observe that a d-cluster tilting module must have all indecomposable injectives and all indecomposable projectives as summands.

Let Λ be d-representation finite. Then we can define the *higher-dimensional Auslander-Reiten translations* by $\tau_d = D\operatorname{Ext}^d_\Lambda(-,\Lambda)$ and $\tau_d^- = \operatorname{Ext}^d_{\Lambda^{op}}(D-,\Lambda)$. We have an analogue of Theorem 2.2.2:

Theorem 2.2.3 ([25, Proposition 1.3])**.** *Let Λ be d-representation finite. Let $P_1,\dots,P_n$ and $I_1,\dots,I_n$ be non-isomorphic representatives of the indecomposable projective respectively injective Λ-modules. Then:*

1. *There is a permutation σ and positive integers $l_1,\dots,l_n$ such that $P_j \cong \tau_d^{l_j-1} I_{\sigma(j)}$ for all j.*
2. *There exists a unique (up to isomorphism) basic d-cluster tilting Λ-module M, given by*

$$M = \bigoplus_{j=1}^{n} \bigoplus_{p=0}^{l_j-1} \tau_d^p I_j = \bigoplus_{j=1}^{n} \bigoplus_{p=0}^{l_j-1} \tau_d^{-p} P_j.$$

3. *The higher Auslander-Reiten translations induce quasi-inverse equivalences*
$$\operatorname{add}(M/P) \underset{\tau_d}{\overset{\tau_d^-}{\rightleftarrows}} \operatorname{add}(M/I),$$
where $P = \bigoplus_{j=1}^n P_j$ *and* $I = \bigoplus_{j=1}^n I_j$.

2.2.3 d-almost split sequences

One key result by Iyama is the existence of a higher-dimensional analogue of almost split sequences.

Theorem 2.2.4 ([24]). *Let Λ be d-representation finite with d-cluster tilting module M, and let $L \in \operatorname{add} M$ be indecomposable and non-injective. Then there exists a unique (up to isomorphism) exact sequence*

$$0 \longrightarrow L \longrightarrow M_d \longrightarrow \cdots \longrightarrow M_1 \longrightarrow N \longrightarrow 0$$

such that

1. *N is indecomposable and all the maps are radical.*
2. *The induced sequence of functors*

$$0 \longrightarrow \operatorname{Hom}_\Lambda(-,L) \longrightarrow \operatorname{Hom}_\Lambda(-,M_d) \longrightarrow \cdots$$

$$\cdots \longrightarrow \operatorname{Hom}_\Lambda(-,M_1) \longrightarrow \operatorname{rad}_\Lambda(-,N) \longrightarrow 0$$

   is exact on $\operatorname{add} M$.

Moreover, in this case $N \cong \tau_d^- L$ and $L \cong \tau_d N$. Such a sequence is also determined by the choice of a non-projective indecomposable N, and the induced sequence

$$0 \longrightarrow \operatorname{Hom}_\Lambda(N,-) \longrightarrow \operatorname{Hom}_\Lambda(M_1,-) \longrightarrow \cdots$$

$$\cdots \longrightarrow \operatorname{Hom}_\Lambda(M_d,-) \longrightarrow \operatorname{rad}_\Lambda(L,-) \longrightarrow 0$$

is also exact on $\operatorname{add} M$.

Such sequences are called *d-almost split sequences*, and as one could expect they play a central role in the study of d-representation finite algebras. Their properties and combinatorial characterisation for tensor products of algebras are the main subject of Papers I and II.

2.3 Preprojective algebras and quivers with potential

In this section we focus on the case $d = 2$ of d-dimensional Auslander-Reiten theory. There is one more construction "in dimension one" which generalises neatly to dimension two, namely the preprojective algebra.

2.3.1 Quivers with potential

For later convenience, before we discuss preprojective algebras we should present the definition of a quiver with potential and its Jacobian algebra [11]. Let Q be a quiver. A potential W on Q is a k-linear combination of cyclic paths in Q, up to cyclic permutations. The pair (Q,W) is called a quiver with potential or a QP for short. Given a potential, one can define an ideal of relations on Q and thus a quotient algebra, called the Jacobian algebra, as follows. For an arrow a in Q, we define the cyclic derivative with respect to a as a linear map $\partial_a : kQ \to kQ$ by setting $\partial_a(p) = \sum_{p=uav} vu$ for every path $p \in kQ$. Observe that $\partial_a(p)$ is 0 unless p is a cycle, in which case it is invariant under cyclic permutation. In particular, the element $\partial_a(W) \in kQ$ is defined. The Jacobian algebra $\mathscr{J}(Q,W)$ is defined by

$$\mathscr{J}(Q,W) = \frac{kQ}{\langle \partial_a(W) \,|\, a \in Q_1 \rangle}.$$

This is not a priori a finite-dimensional algebra, and in the generality we defined it the ideal we quotient by is not necessarily admissible. In particular, a necessary condition is that all cycles appearing in W have length at least three. It is often more convenient to consider the completed version of the Jacobian algebra, where the difference is that we allow for infinite sums in the path algebra and for potentials to be infinite linear combinations of cycles.

Jacobian algebras were introduced (to algebraists) in [11] to connect representation theory and cluster theory (i.e. the study of phenomena related to the cluster algebras of [12]). In fact, if T is a (2-)cluster tilting object in a suitable category, it often happens that $\mathrm{End}(T)$ is a Jacobian algebra. It is therefore not surprising that quivers with potential and their Jacobian algebras have been widely and successfully used to categorify cluster algebras in various contexts [1] [8] [32]. Jacobian algebras also make an appearance in 2-dimensional Auslander-Reiten theory as preprojective algebras, as we will now explain.

2.3.2 The (classical) preprojective algebra

If Δ is a simply laced Dynkin diagram and Q is a quiver with underlying graph Δ, there is a way to associate a finite-dimensional algebra $\Pi(Q)$ to Q that, up to isomorphism, depends only on Δ. Moreover, Q can be recovered from $\Pi(Q)$ with the datum of a certain grading corresponding to the choice of an orientation of Δ. The construction is as follows: let us define the double quiver $\overline{Q}$ of Q by $\overline{Q}_0 = Q_0$ and $\overline{Q}_1 = Q_1 \cup \{\bar{a} : j \to i \,|\, a : i \to j \in Q_1\}$. Observe that, as a quiver, $\overline{Q}$ only depends on Δ and not on the orientation Q. We define the preprojective algebra $\Pi(Q)$ by

$$\Pi(Q) = \frac{k\overline{Q}}{\left\langle e_i \left(\sum_{a\in Q_1} a\bar{a} - \bar{a}a\right) e_i \,|\, i \in Q_0 \right\rangle}.$$

This definition was introduced in [16], and the algebra $\Pi(Q)$ turns out to be finite-dimensional and self-injective. Moreover, if Q' is another quiver with underlying graph Δ, then there is an isomorphism $\Pi(Q) \cong \Pi(Q')$. Another definition was later given in [6]: one can define $\Pi(Q)$ to be the tensor algebra

$$\Pi(Q) = \bigoplus_{i\geq 0} \mathrm{Ext}^1_{kQ}(D(kQ), kQ)^{\otimes i}.$$

One can recover the path algebra kQ in the following way. We can define a grading on $k\overline{Q}$ by setting all arrows $a \in Q_1$ to have degree zero, and all arrows $\bar{a}$ to have degree one. Then the relations defining $\Pi(Q)$ are homogeneous, so we obtain an induced grading on $\Pi(Q)$. One can check that we can then recover kQ as the degree zero part $kQ \cong \Pi(Q)_0$. Note that all orientations of Δ appear in this way for a suitable choice of grading.

2.3.3 The 3-preprojective algebra

This construction generalises neatly, mutatis mutandis, to global dimension two. A suitable 2-dimensional analogue of representation-finite path algebras (i.e. path algebras of Dynkin quivers) is given by 2-representation finite algebras. If Λ is 2-representation finite, one can define [27] the (3-)preprojective algebra $\Pi(\Lambda)$ to be

$$\Pi(\Lambda) = \bigoplus_{i\geq 0} \mathrm{Ext}^2_{\Lambda}(D\Lambda, \Lambda)^{\otimes i}.$$

This is again finite-dimensional and self-injective, and moreover it was shown by Keller [31] that, if k is algebraically closed, there exists a QP (Q,W) such that $\Pi(\Lambda) \cong \mathscr{J}(Q,W)$.

Like in the 1-dimensional case, there are many 2-representation finite algebras sharing the same preprojective algebra, and one can recover them all by a suitable choice of grading. Specifically, a *cut* C on a QP (Q,W) is a set of arrows of Q such that every cycle of W has exactly one arrow in C. One can then define a grading on kQ by setting arrows in C to have degree one and all the other arrows to have degree zero. Since the potential is homogeneous by definition, one gets a grading on the Jacobian algebra, and one can consider the degree zero part $\mathscr{J}(Q,W)_C$, called a truncated Jacobian algebra.

Theorem 2.3.1 ([19]). *Let $\mathscr{J}(Q,W)$ be a self-injective Jacobian algebra, such that the ideal $\langle \partial_a(W) \,|\, a \in Q_1 \rangle$ defining it is admissible. If C is a cut, then $\mathscr{J}(Q,W)_C$ is 2-representation finite. In this case,*

$$\mathscr{J}(Q,W) \cong \Pi(\mathscr{J}(Q,W)_C).$$

Moreover, if k is algebraically closed, every basic 2-representation finite algebra can be described in this way for some cut C on the QP (Q,W) of its preprojective algebra.

This result motivates the investigation of self-injective Jacobian algebras (and of their cuts). We carry out such an investigation in Papers III, IV and V. We are able to draw from different areas of algebra to construct many new examples of self-injective Jacobian algebras, and in some cases to construct cuts on them. In the following, we say that a QP is self-injective if its Jacobian algebra is self-injective.

3. Summary of papers

In the core of this thesis we address various problems and constructions related to higher-dimensional Auslander-Reiten theory. In this chapter, we summarise the main results contained in the papers of which this core consists. The themes covered are tensor products (Papers I and II), Postnikov diagrams and quivers with potential (Papers III, IV and V) and skew group algebras (Paper V). Paper I deals with the question of describing $(n+m)$-almost split sequences over a tensor product of an n- and an m-representation finite algebra, when this is known to be $(n+m)$-representation finite. In Paper II we extend the results and constructions of Paper I to the weaker setting of d-complete algebras. We prove that if A and B are acyclic n- respectively m-complete algebras, then $A \otimes B$ is acyclic $(n+m)$-complete. In Paper III we study QPs constructed combinatorially from Postnikov diagrams, and prove that they are self-injective if and only if the diagram is rotation invariant. Motivated by this, we investigate in Paper IV for which parameters there exist rotation-invariant Postnikov diagrams, and find a necessary and sufficient condition. Paper V is dedicated to the study of skew group algebras of Jacobian algebras, and of how one can translate into combinatorial operations on the QP level the algebraic construction of taking skew group algebras. We obtain various results, which we apply in particular to self-injective QPs coming from Postnikov diagrams.

3.1 Paper I

The first construction we address in this thesis is that of tensor product. If A and B are k-algebras, one can take their tensor product (as vector spaces) $\Lambda = A \otimes_k B$, and define multiplication componentwise to get a k-algebra. To ensure that homological algebra behaves well, we assume that k is perfect (in particular, this guarantees that $\operatorname{gl.dim}(\Lambda) = \operatorname{gl.dim}(A) + \operatorname{gl.dim}(B)$). Let A and B be an n- and an m-representation finite algebra. In general it is not true that Λ is $(n+m)$-representation finite, but there is a necessary and sufficient condition for when this happens, found in [18]. The question we investigate in Paper I is:

Question 3.1.1. *Suppose that A, B and Λ are n-, m- and $(n+m)$-representation finite respectively. What is the connection between the higher almost split sequences in* $\operatorname{mod} A$, $\operatorname{mod} B$ *and* $\operatorname{mod} \Lambda$*?*

One naive guess would be to say that higher almost split sequences over Λ are total tensor products of higher almost split sequences over A and B. This cannot be true, however, since the total tensor product of complexes of length $n+2$ and $m+2$ has length $n+m+3$, while $(n+m)$-almost split sequences have length $n+m+2$.

A description of the $(n+m)$-cluster tilting subcategory $\mathscr{C}_\Lambda$ of Λ, as well as of the $(n+m)$-Auslander-Reiten translations, was obtained in [18]. It turns out that

$$\mathscr{C}_\Lambda = \operatorname{add}\left(\bigoplus_{i\geq 0} \tau_n^{-i} A \otimes \tau_m^{-i} B\right).$$

Moreover, for every indecomposable $N \otimes M \in \mathscr{C}_\Lambda$ we have $\tau_{n+m}^{\pm}(N \otimes M) \cong \tau_n^{\pm}(N) \otimes \tau_m^{\pm}(M)$. In particular, we have that $\tau_n^-(N) \neq 0$ and $\tau_m^-(M) \neq 0$ precisely if $\tau_{n+m}^-(N \otimes M) \neq 0$. We define *slices*: slice i of $\operatorname{mod}\Lambda$ is the full subcategory of $\operatorname{mod}\Lambda$ given by $\mathscr{S}_\Lambda(i) = \operatorname{add}(\tau_{n+m}^{-i}\Lambda)$. In a similar way we define slices of $\operatorname{mod}A$ and $\operatorname{mod}B$, so that $\otimes$ gives a map $\mathscr{S}_A(i) \times \mathscr{S}_B(i) \to \mathscr{S}_\Lambda(i)$.

Assume that $N \otimes M \in \mathscr{C}_\Lambda$ is the starting point of an $(n+m)$-almost split sequence. Then there is i such that $N \in \mathscr{S}_A(i)$, $M \in \mathscr{S}_B(i)$ and $N \otimes M \in \mathscr{S}_\Lambda(i)$. Moreover, N is the starting point of an n-almost split sequence and M of an m-almost split sequence. Describing the $(n+m)$-almost split sequences in $\operatorname{mod}\Lambda$ can be seen as completing the information we have about the category $\mathscr{C}_\Lambda$: given the n-almost split sequences starting in N and the m-almost split sequence starting in M, can we describe the $(n+m)$-almost split sequence starting in $N \otimes M$ (when $N \otimes M \in \mathscr{C}_\Lambda$)?

It turns out that the tool we need to describe the operation of "tensoring" an n- and an m-almost split sequence is given by the mapping cone. Given a chain map $\varphi : (A_\bullet, d^A) \to (B_\bullet, d^B)$, its *mapping cone* is the complex $\operatorname{Cone}(\varphi) = A[-1]_\bullet \oplus B_\bullet$ with differential given by

$$d^{\operatorname{Cone}(f)} = \begin{pmatrix} d^A[-1] & 0 \\ \varphi[-1] & d^B \end{pmatrix}.$$

In Paper I we prove:

Theorem 3.1.2. *Let Λ be a d-representation finite algebra. Every d-almost split sequence in* $\operatorname{mod}\Lambda$ *is isomorphic as a complex to the mapping cone of a chain map.*

The chain maps giving d-almost split sequences are of the form $\varphi : C_1 \to C_2$, where C_1 and C_2 are complexes consisting of modules in slice i and $i+1$ respectively, for some i. There is a natural notion of tensor product of chain maps, so one could ask whether $(n+m)$-almost split sequences can be realised as cones of tensor products of maps. The main result of Paper I is that this is indeed the case, thus giving a complete description of higher almost split sequences over a tensor product in terms of the ones over the factors.

Theorem 3.1.3. *Let A, B and $\Lambda = A \otimes B$ be n-, m- and $(n+m)$-representation finite respectively. Let φ be a chain map from $\mathscr{S}_A(i)$ to $\mathscr{S}_A(i+1)$ such that* $\operatorname{Cone}(\varphi)$ *is n-almost split, and let ψ be a chain map from $\mathscr{S}_B(i)$ to $\mathscr{S}_B(i+1)$ such that* $\operatorname{Cone}(\psi)$ *is m-almost split. Then* $\operatorname{Cone}(\varphi \otimes \psi)$ *is $(n+m)$-almost split.*

Since every $(n+m)$-almost split sequence starts in $\mathscr{S}_\Lambda(i) = \mathscr{S}_A(i) \otimes \mathscr{S}_B(i)$ for some i, all $(n+m)$-almost split sequences in $\operatorname{mod}\Lambda$ are accounted for by this theorem.

3.2 Paper II

The idea behind Paper II is to take the constructions of Paper I and perform them in greater generality. In particular, we consider a setting in which slices can be defined and we have $\mathscr{S}_{A\otimes B}(i) = \mathscr{S}_A(i) \otimes \mathscr{S}_B(i)$, but we do not have d-representation finiteness.

Question 3.2.1. *We know that* $\operatorname{Cone}(\varphi \otimes \psi)$ *will have good homolgical properties whenever* $\operatorname{Cone}(\varphi)$ *and* $\operatorname{Cone}(\psi)$ *are n- and m-almost split. This does not depend on the fact that $A \otimes B$ is $(n+m)$-representation finite. Can we still say something about nice subcategories of* $\operatorname{mod} A \otimes B$ *based on the existence of such sequences?*

We consider, instead of d-representation finite algebras, the weaker notion of d-complete algebras. Let Λ be d-representation finite. Recall that, by Theorem 2.2.3, for every indecomposable injective Λ-module I there is $l_I \geq 1$ such that $\tau_d^{l_I - 1}(I)$ is projective. Moreover, the d-cluster tilting subcategory is given by

$$\mathscr{C}_\Lambda = \operatorname{add}\left(\bigoplus_{I \text{ ind. injective}} \bigoplus_{j=0}^{l_I - 1} \tau_d^j(I) \right).$$

Observe in particular that $\tau_d^{l_I}(I) = 0$. We want to generalise this setup, but without the condition of $\tau_d^{l_I - 1}(I)$ being projective.

Definition 3.2.2. A Λ-module T is a *tilting module* if:

1. $\operatorname{proj.dim} T \leq 1$.

2. $\mathrm{Ext}^i_\Lambda(T,T) = 0$ for all $i > 0$.

3. There is an exact sequence

$$0 \longrightarrow \Lambda \longrightarrow T_0 \longrightarrow \cdots \longrightarrow T_m \longrightarrow 0$$

for some m, with $T_i \in \operatorname{add} T$ for all i.

Intuitively, Λ is d-complete if, instead of reaching the indecomposable projectives as the last nonzero translates of the indecomposable injectives, we reach a suitable tilting module. In this case the category $\mathscr{C}_\Lambda$ defined above will not be d-cluster tilting in $\operatorname{mod}\Lambda$, but only in the exact subcategory $T^\perp = \ker \mathrm{Ext}^{>0}(T,-)$. More precisely, we call $\mathscr{P} = \{X \in \mathscr{C}_\Lambda \mid \tau_d X = 0\}$, $\mathscr{C}_P = \{X \in \mathscr{C}_\Lambda \mid X \text{ has no nonzero summands in } \mathscr{P}\}$, and T a basic module such that $\operatorname{add} T = \mathscr{P}$. We define:

Definition 3.2.3. An algebra Λ of global dimension at most d is *d-complete* if the following conditions hold:

1. T is a tilting module.

2. $\mathscr{C}_\Lambda$ is a d-cluster tilting subcategory of $T^\perp$,

3. $\mathrm{Ext}^i(\mathscr{C}_P, \Lambda) = 0$ for every $0 < i < d$.

So d-representation finite is the same as d-complete with $T = \Lambda$. The notion of d-completeness was originally introduced in [25] to deal with higher Auslander algebras. If Λ is d-representation finite, one can define its *d-Auslander algebra* as the endomorphism algebra of the basic d-cluster tilting Λ-module. For $d = 1$ this coincides with the classical notion of the Auslander algebra of a representation finite algebra. One of Iyama's main motivations for introducing higher-dimensional Auslander-Reiten theory was to obtain a generalisation of the Auslander correspondence, which describes a necessary and sufficient homological condition for an algebra to be an Auslander algebra.

If Λ is d-representation finite, its d-Auslander algebra has global dimension at most $d+1$, but it is not usually $(d+1)$-representation finite. However, Iyama proved that if Λ is d-complete, then its d-Auslander algebra is always $(d+1)$-complete. As we saw, if A and B are n- respectively m-representation finite, the tensor product $A \otimes B$ is not $(n+m)$-representation finite in general. In Paper II we prove:

Theorem 3.2.4. *Let A and B be n- respectively m-complete acyclic algebras. Then $A \otimes B$ is $(n+m)$-complete and acyclic.*

This theorem can be seen as a parallel to Iyama's result about higher Auslander algebras, in that we find one more setting in which d-representation finiteness is too strong a property to be preserved, but d-completeness is not.

The assumption of acyclicity is a technical condition explained in Definition 2.5 and §4.4 of Paper II. If $\Lambda = kQ/I$ for an admissible ideal I and k is algebraically closed, then Λ is acyclic if and only if Q is acyclic. We need the assumption that A and B be acyclic in order to make proofs work, but we remark that there are no known examples of d-complete algebras which are not acyclic.

In particular, we get back the characterisation, originally found in [18], of when the tensor product of an n- and an m-representation finite algebra is $(n+m)$-representation finite (in the acyclic case). We say that an algebra Λ of global dimension d is *l-homogeneous* if $\tau_d^{l-1}(D\Lambda) = T$. If Λ is d-complete, this is the same as saying that $l_I = l$ for every indecomposable injective I. We get:

Corollary 3.2.5. *Let A and B be n- respectively m-representation finite acyclic algebras. Then the following are equivalent:*

1. *$A \otimes B$ is $(n+m)$-representation finite;*
2. *A and B are l-homogeneous for some common l.*

Moreover, in this case $A \otimes B$ is also l-homogeneous.

To prove that $A \otimes B$ is $(n+m)$-complete, we first prove that $\operatorname{mod} A \otimes B$ has $(n+m)$-almost split sequences using the same method as in Paper I. Namely, we realise n- and m-almost split sequences over A and B as cones of chain maps, and then verify that the cone of the tensor product is indeed $(n+m)$-almost split. Then, by a recursive construction using these sequences (and making crucial use of acyclicity) we can prove that T is a tilting module in $\operatorname{mod}\Lambda$, which then implies d-completeness using a theorem by Iyama. The difficult part is precisely showing that T is tilting, since the "generating" property needs in principle to be checked for arbitrary $A \otimes B$-modules, and not only for modules of the form $N \otimes M$. Instead, we use an argument involving an alternative generating property of tilting modules, namely that they generate the bounded derived category.

3.3 Paper III

The motivation behind Paper III is: there is a certain combinatorial way of generating planar QPs in the sense of [19]. Can we use it to produce self-injective QPs? Recall that we are particularly interested in self-injective QPs because their Jacobian algebras are the 3-preprojective algebras of 2-representation finite algebras.

The combinatorics is that of Postnikov diagrams [36] [7]. A (k,n)-Postnikov diagram (in this paper k is a number and not a field) is a collection of n oriented curves in a disk with n marked points on the boundary. The curves connect vertex i to vertex $i+k \pmod{n}$, and the key property is that following one curve one sees the others crossing it alternatingly from the left and from the right. To such a diagram one can associate a quiver by putting vertices in the regions whose boundary is alternating, and connecting them with arrows through the crossings. This is illustrated in Figure 3.1.

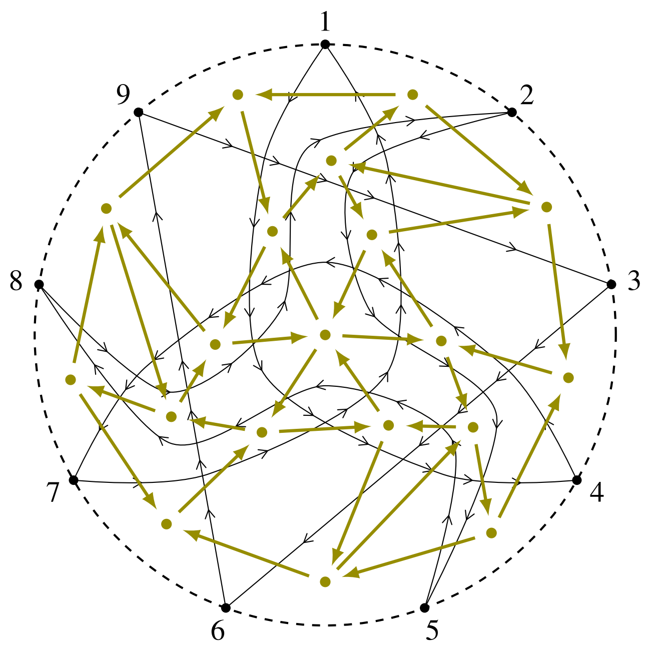

Figure 3.1. A rotation-invariant $(3,9)$-Postnikov diagram and the corresponding quiver.

One obtains a planar QP (by setting the potential to be the sum of the boundaries of the faces) whose Jacobian algebra is infinite dimensional. Factoring out the idempotent corresponding to the boundary vertices, one gets on the other hand a finite-dimensional Jacobian algebra, and it makes sense to ask whether this is self-injective. In practice, we just remove the boundary vertices, so from the quiver of Figure 3.1 we obtain the planar QP of Figure 3.2.

Question 3.3.1. *What kind of Jacobian algebras arise in this way? Can we characterise their self-injectivity?*

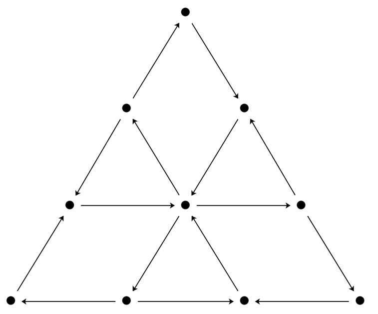

Figure 3.2. The planar self-injective QP corresponding to the Postnikov diagram of Figure 3.1.

In all the examples of self-injective planar QPs shown in [19], a Nakayama automorphism acts by a rotation, and it turns out that the same is true here. The main result of Paper III is:

Theorem 3.3.2. *The Jacobian algebra constructed from a (k,n)-Postnikov diagram is self-injective if and only if the diagram is invariant under rotation by $\frac{2\pi k}{n}$. In this case, a Nakayama automorphism acts by this rotation.*

Observe that one can interpret the action of a Nakayama automorphism as subtracting $k \pmod n$ from all labels, a fact which plays a role in the proof (as well as in Paper IV).

The proof uses a categorification of Postnikov diagrams constructed in [29]. There is an algebra $B = B(k,n)$ such that the combinatorics of (k,n)-Postnikov diagrams governs the vanishing of Ext^1 between certain Cohen-Macaulay B-modules. One can define a B-module L_I for each vertex I of the quiver associated to the diagram, and it turns out that $T = \bigoplus_I L_I$ is a cluster tilting object in the stable category $\underline{\mathrm{CM}}(B)$. By a result of [7], the stable endomorphism algebra $\underline{\mathrm{End}}_B(T)$ is in fact isomorphic to the Jacobian algebra associated to the Postnikov diagram. Moreover, $\underline{\mathrm{CM}}(B)$ is 2-Calabi-Yau, which implies that $\underline{\mathrm{End}}_B(T)$ is a self-injective algebra if and only if $T \cong T[2]$. We compute the action of the functor $[2]$ on the modules L_I, and prove that $T \cong T[2]$ precisely when the Postnikov diagram is rotation invariant.

We also consider cuts on QPs coming from Postnikov diagrams. Using an isoradial embedding constructed in [35], we show:

Proposition 3.3.3. *If (Q,W) is a QP constructed from a Postnikov diagram, then every arrow of Q is contained in a cut.*

For planar QPs, the condition that every arrow be contained in a cut was introduced and studied in [19]. In particular, Herschend and Iyama prove that if it holds, then all the truncated Jacobian algebras $\mathscr{J}(Q,W)_C$ are derived equivalent.

Paper III also contains some new examples of self-injective QPs, constructed from Postnikov diagrams. This answers in the negative a question asked in [19]. In particular, a family of planar self-injective QPs with Nakayama automorphism of arbitrarily large order is constructed.

3.4 Paper IV

A natural question to ask in view of the results of Paper III is:

Question 3.4.1. *For which pairs (k,n) do there exist rotation-invariant (k,n)-Postnikov diagrams?*

If we do not ask for rotation invariance, the answer is for all $n \geq k \geq 1$. If we do, however, the situation is more complicated. In Paper IV we answer this question:

Theorem 3.4.2. *There exists a rotation-invariant (k,n)-Postnikov diagram if and only if k is congruent to $0,1$ or -1 modulo $n/\operatorname{GCD}(k,n)$.*

In this paper we use the language of *maximal noncrossing collections* instead of that of Postnikov diagrams. Two subsets I and J of $\{1,\dots,n\}$ are said to be *noncrossing* if there do not exist cyclically ordered a,b,c,d such that $a,c \in I \setminus J$ and $b,d \in J \setminus I$. To a Postnikov diagram one can associate a collection of mutually noncrossing k-element subsets of $\{1,\dots,n\}$ by assigning to every region with alternating boundary the set of starting points of curves that have the region to their left (see Figure 3.1).

By results in [36] and [35], the resulting collection of k-element sets is maximal among the collections of mutually noncrossing k-element sets, and we call it a maximal noncrossing collection. Moreover, all maximal noncrossing collections arise in this way [35], so Postnikov diagrams and maximal noncrossing collections are interchangeable as combinatorial objects. We choose here to work with collections because they are easier to construct explicitly. The rotation invariance can be phrased in this language by demanding that the collection be invariant under adding $k \pmod n$ to all labels (we call such collections *symmetric*).

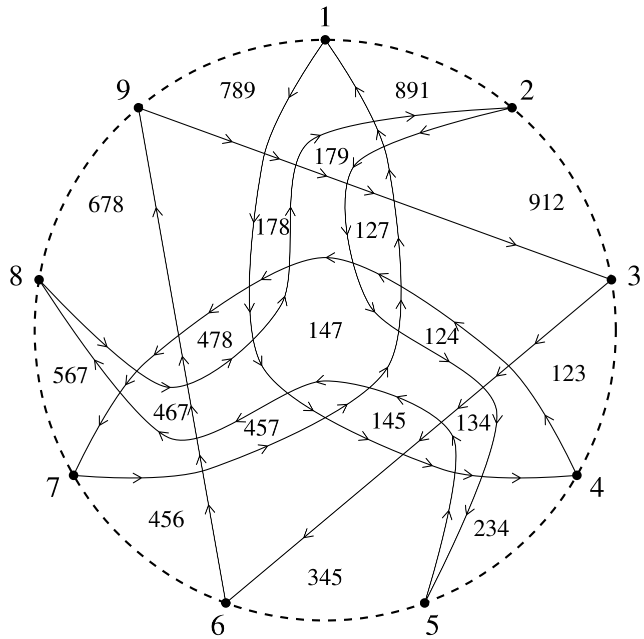

Figure 3.3. The maximal noncrossing collection associated to the Postnikov diagram of Figure 3.1.

The two implications in the theorem are proved with different techniques. Necessity of the numerical condition follows from analysing the isoradial embedding (see [35]) of the quiver associated to a Postnikov diagram. This quiver must have a central vertex or a central cycle in order to be rotation invariant, and the condition follows from considerations on its associated k-element set(s).

Sufficiency of the numerical condition is proved by explicitly constructing a symmetric maximal noncrossing collection when the condition holds. This is the core of the paper, and it makes crucial use of the fact that all maximal noncrossing collections of k-element subsets of $\{1,\dots,n\}$ have exactly $k(n-k)+1$ elements [35]. It is worth noting that we do not get all the possible symmetric maximal noncrossing collections this way, so a complete classification is still open.

Recall that in Paper III we established that the Jacobian algebras constructed from rotation-invariant Postnikov diagrams are self-injective, with a Nakayama automorphism induced by rotation. A corollary of our result is therefore:

Corollary 3.4.3. *There exist infinitely many self-injective Jacobian algebras with Nakayama automorphism of any prescribed order.*

3.5 Paper V

Paper V deals with yet another construction, namely that of skew group algebra. If Λ is a k-algebra and G is a finite group acting on Λ by algebra automorphisms, one can define the skew group algebra ΛG to be $\Lambda \otimes_k kG$ as a vector space, with the "twisted" multiplication induced by

$$(\lambda \otimes g)(\mu \otimes h) = \lambda g(\mu) \otimes gh.$$

The algebra ΛG is not basic in general, so it is not the quotient of a path algebra. However, it is interesting to study ΛG up to Morita equivalence, in particular the connections between the quiver of Λ and that of ΛG. This was done extensively by Reiten and Riedtmann [37], who proved that many properties of $\operatorname{mod} \Lambda G$ are inherited from $\operatorname{mod} \Lambda$. One interesting question is to describe the quiver Q_G of a basic version Λ_G of ΛG explicitly as a function of the quiver of Λ. This was done in [37] if G is cyclic, and in [10] for any G if Λ is hereditary or a preprojective algebra. However, describing the relations one needs to impose on Q_G to obtain Λ_G is difficult in general. In this paper we address the following question:

Question 3.5.1. *Let (Q,W) be a QP with a group G acting on it in a nice way, and let $\Lambda = \mathscr{J}(Q,W)$. Is the skew group algebra ΛG Morita equivalent to another Jacobian algebra $\mathscr{J}(Q_G,W_G)$? If it is, can we explicitly describe the potential W_G?*

A positive answer to the first part of this question comes from work by Le Meur [33]. He proves that the skew group dg algebra of the *Ginzburg dg algebra* of (Q,W) is Morita equivalent to the Ginzburg dg algebra of another QP. Then one can get the statement for Jacobian algebras by taking zeroth cohomology. The potential one obtains on Q_G is the image of W via a natural map, but it is not expressed as a linear combination of cycles. By a similar approach, Amiot and Plamondon [3] manage to describe W_G explicitly if $G \cong \mathbb{Z}/2\mathbb{Z}$.

In the first part of Paper V, we explicitly describe the potential W_G under some assumptions on the action of G on (Q,W). We work in the case where G is cyclic, and we impose some combinatorial conditions on the length of the orbits of vertices in function of W. The precise statement we prove is:

Theorem 3.5.2. *Let (Q,W) be a QP and Λ its Jacobian algebra. Let G be a finite cyclic group acting on (Q,W) as per the assumptions (A1)–(A7) of §3.1 of Paper V. Let $\eta \in \Lambda G$, Q_G and W_G be as in Section 3 of Paper V. Then*

$$\mathscr{J}(Q_G,W_G) \cong \eta(\Lambda G)\eta.$$

These conditions (A1)–(A7) are satisfied in many examples, namely whenever (Q,W) is a planar QP on which G acts by rotations. Thus we get many examples coming from self-injective Jacobian algebras constructed in Papers III and IV (one can take G generated by a power of the Nakayama automorphism).

Reiten and Riedtmann observe that, in all the examples they compute, there is a natural action of the dual group $\hat{G}$ of G on Q_G. They prove that in fact there is always a natural action of $\hat{G}$ on ΛG, and that if G is abelian then the algebra $(\Lambda G)\hat{G}$ is Morita equivalent to Λ. Thus taking skew group algebras is in some sense a duality in the abelian case. In the second part of the paper, we prove that in our setting the action of $\hat{G}$ on ΛG restricts to the basic version by an action on (Q_G, W_G). This action still satisfies our assumptions, so we can explicitly construct a QP $((Q_G)_{\hat{G}}, (W_G)_{\hat{G}})$ whose Jacobian algebra is isomorphic to Λ. We prove that, as one could expect,

Theorem 3.5.3. *There is an isomorphism of QPs*

$$((Q_G)_{\hat{G}}, (W_G)_{\hat{G}}) \cong (Q,W)$$

which induces an isomorphism of algebras

$$\theta \left((\eta \, (\Lambda G) \, \eta) \, \hat{G} \right) \theta \cong \Lambda,$$

where θ is the idempotent defined in Section 5 of Paper V.

The third part of Paper V is dedicated to studying planar QPs with G acting by rotations and their skew group algebras. We construct cuts on Q_G from G-invariant cuts on Q, and show some conditional results about the truncated Jacobian algebras of ΛG.

4. Afterword

Strenuousness is the immortal path, sloth is the way of death.

—H. W. Tilman, When Men & Mountains Meet

Only by getting to the top does one realise that there are so many mountains around. The results in this thesis are the product of a long process, during which many questions were answered. Fortunately, many were left unanswered, and many more new questions were asked. The aim of this chapter is to outline some possible directions for future (and present) research.

4.1 Generalising Postnikov diagrams

In the scope of the main result of Paper III, there is still a lot to be done. Suppose we construct a Postnikov diagram such that the permutation afforded by the strands is not $i \mapsto i+k \pmod{n}$, but which is symmetric. This corresponds to a collection of noncrossing subsets which is maximal in a suitable submatroid of the matroid of all k-element subsets of $\{1,\dots,n\}$ [35]. The QP one gets is still rotation-invariant, but not always self-injective. Yet, all examples of planar self-injective QPs that I know which have no interior vertices of valency 2 can be realised in this way. Are there others? Exactly what conditions do we need for self-injectivity?

Here is a proposed strategy for looking at these questions. Let $\mathbb{J}$ be a noncrossing collection which is maximal in some suitable matroid. One should associate an algebra B' to $\mathbb{J}$ in a similar way as how B is constructed, and there should exist a map $B \to B'$ (corresponding to the fact that $\mathbb{J}$ can be embedded in a collection which is actually maximal). By studying properties of this map and the corresponding functors $\operatorname{mod} B' \to \operatorname{mod} B$, one can hope to obtain information about self-injectivity of the endomorphism algebra of the (conjectural) cluster tilting B'-module associated to $\mathbb{J}$. In particular, I hope that the restriction of scalars functors $\mathrm{CM}(B') \to \mathrm{CM}(B)$ behave reasonably with respect to the triangulated structures of the stable categories.

4.2 More on skew group algebras

One problem with the main result of Paper V is the dependence on many assumptions on both the QP and the group action. These assumptions are satisfied in the cases we were interested in, but are indeed quite strong. One could try to generalise the result to the case of an arbitrary group acting reasonably on an arbitrary QP.

This was done in [34], using techniques from monoidal categories. However, the formulas Le Meur obtains depend on solving a linear system which can be quite big. In an ongoing joint project with Giovannini and Plamondon, we are trying to give simple formulas for the potential W_G for any abelian group G. In fact, we hope to obtain a slightly stronger result, namely an isomorphism not only on the level of Jacobian algebras but also on the level of dg algebras (this is the generality of [3] [33] [34], but not of Paper V).

4.3 Postnikov diagrams on orbifolds

We observed in Paper V how one can always apply our skew group QP construction to the self-injective QPs coming from Postnikov diagrams. On the one hand, one could try to find a combinatorial model in the spirit of Postnikov diagrams for the resulting QPs. This should be axiomatised as some kind of strand diagram on the disk with an orbifold point. On the other hand, one can construct the skew group category of $\mathrm{CM}(B)$, and this will be a Frobenius, stably 2-Calabi-Yau category. One could imagine it being equivalent to $\mathrm{CM}(B_G)$, where B_G is the skew group algebra of B. Furthermore, there is hope to describe cluster tilting modules in $\mathrm{CM}(B_G)$ combinatorially, and maybe one can recover a “skew” version of [7, Theorem 10.3]. It is an ongoing project with Baur and Velasco to investigate these questions.

5. Sammanfattning på svenska (Summary in Swedish)

Den här avhandlingen består av fem artiklar om representationsteori av algebror. I det här kapitlet kommer vi att återge en del av den algebraiska bakgrunden till avhandlingens resultat, och sedan att sammanfatta själva resultaten. För fördjupningar inom algebra och speciellt representationsteori hänvisar vi till [17] och [4].

5.1 Bakgrund

En algebra är en mängd där man kan addera och multiplicera ihop element, samt multiplicera element med skalärer (till exempel reella tal). Ett sätt att beskriva en algebra är att presentera den som vägalgebran av ett koger, modulo några relationer. Ett koger är en uppsättning punkter med en uppsättning pilar mellan dem. Vägalgebran är mängden av alla summor av alla skalärmultiplar av riktade vägar i ett koger, med multiplikation given av sammansättning av konsekutiva vägar. Vägalgebran kan modifieras genom att kvota bort en del relationer, vilket innebär att vissa linjära kombinatoner av vägar blir lika med noll. Alla basala ändligdimensionella algebror över en algebraiskt sluten kropp kan skrivas som kvot av en vägalgebra på det sättet.

Representationsteori handlar om att beskriva moduler över algebror. Dessa är vektorrum där algebran agerar som en mängd av endomorfier. Problemet att beskriva modulkategorin av en algebra är generellt olösbart, men man kan byta till enklare, mer specifika problem. En idé är att beakta endast en delkategori om vilken man kan säga någonting. Det är motivationen bakom högdimensionell Auslander-Reitenteori, som uppfanns under de senaste 15 år av Iyama och hans medarbetare [24].

På ett koger kan man ange en potential, det vill säga en linjärkombination cykler. En potential ger upphov till vissa relationer och på det sättet fås en så kallad Jacobialgebra [11].

5.2 Avhandlingens resultat

I Artikel I och II betraktar vi högredimensionell Auslander-Reitenteori för tensorprodukten $\Lambda = A \otimes B$ av en n- och en m-representationsändlig algebra, se också [18]. Resultatet är en algebra som vanligtvis inte är $(n+m)$-representationsändlig, men vi antar att den ar det i Artikel I. Här undersöker vi formen på de så kallade d-nästan kluvna följderna i $\operatorname{mod}\Lambda$. Vi bevisar att beroendet på A och B kan beskrivas med hjälp av total tensorprodukt, en klassisk homologisk konstruktion. De n- respektive m-nästan kluvna följderna hos A respektive B kan realiseras som koner av vissa avbildningar φ respektive ψ. Med hjälp av total tensorprodukt fås en avbildning $\varphi \otimes \psi$ vars kon ger den sökta $(n+m)$-nästan kluvna följden.

I Artikel II används den här konstruktionen i ett mer generellt sammanhang, nämligen för d-fullständiga algebror [25]. Vi bevisar att tensorprodukten $\Lambda = A \otimes B$ av en m- och en n-fullständig algebra är $(n+m)$-fullständig (under ett visst antagande som stämmer i alla kännda exempel). Formlerna i Artikel I används för att visa existensen av $(n+m)$-nästan kluvna följder som sedan leder till $(n+m)$-fullständighet av Λ.

Artikel III adresserar en annan fråga. Den handlar om att konstruera nya exempel på Jacobialgebror av självinjektiva koger med potential. De här algebrorna spelar en viktig roll inom 2-dimensionell Auslander-Reitenteori [19]. Metoden vi använder utnyttjar vissa uppsättning kurvor på enhetscirkelskivan, så kallade Postnikovdiagram. I artikeln bevisar vi att en viss rotationsymmetri hos diagrammet är nödvandig och tillräcklig för algebrans självinjektivitet. Beviset använder en kategorifiering av Postnikovdiagram med moduler över en viss oändligdimensionell algebra. Denna algebra infördes i [29] för att kategorifiera den Grassmannska klusteralgebran. I kategorifieringen motsvarar rotationsymmetri en viss algebraisk invarians hos modulerna, som leder till självinjektivitet.

I Artikel IV undersöker vi symmetriska Postnikovdiagrams existens. Ett sådant diagram beror på två parametrar (k,n), och symmetrivillkoret är inte möjligt för ett godtyckligt val av dem. Vi bevisar att ett nödvandigt och tillräckligt villkor på parametrarna är $k \equiv -1, 0$ eller 1 modulo $n/\operatorname{SGD}(k,n)$. Beviset är konstruktivt, det vill säga vi skapar ett explicit symmetriskt Postnikovdiagram i alla fall där ett finns.

Artikel V undersöker en till konstruktion, nämligen skevgruppsalgebror. De är algebror definierade av en gruppverkan på en algebra. Om den ursprungliga algebran var en Jacobialgebra, vet vi tack vare [33] att dess skevgruppsalgebra också blir (Moritaekvivalent till) en Jacobialgebra. Dessutom kan skevgruppsalgebrans koger beskrivas fullständigt [10] [37]. Å andra sidan, är det i allmänhet svårt att beskriva den nya potentialen på ett explicit sätt. Det är det vi gör i artikeln, under vissa antaganden: vi får formler för potentialen för en Jacobialgebras skevgruppsalgebra. Antagandena stämmer alltid om vi betraktar en Jacobialgebra som kommer från ett Postnikovdiagram. Detta, kombinerat med Artikel III och IV:s resultat, ger oss en rik källa av självinjektiva och symmetriska Jacobialgebror.

Det finns dessutom en viss dualitet hos konstruktionen skevgruppsalgebra. Den duala gruppen verkar på skevgruppsalgebran, och vi bevisar att under våra antaganden blir själva skevgruppsalgebrans skevgruppsalgebra samma som (Moritaekvivalent till) den ursprungliga algebran.

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Paper I

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Tensor products of higher almost split sequences

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ABSTRACT

We investigate how the higher almost split sequences over a tensor product of algebras are related to those over each factor. Herschend and Iyama give in [6] a criterion for when the tensor product of an n-representation finite algebra and an m-representation finite algebra is $(n+m)$-representation finite. In this case we give a complete description of the higher almost split sequences over the tensor product by expressing every higher almost split sequence as the mapping cone of a suitable chain map and using a natural notion of tensor product for chain maps.

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1. Introduction and conventions

In the context of Auslander–Reiten theory one can study almost split sequences of modules over a finite-dimensional algebra A. These are certain short exact sequences

$$0 \to M \to N \to L \to 0$$

such that M and L are indecomposable, and it turns out that every nonprojective indecomposable module over A appears as the last term of such a sequence (and every noninjective indecomposable appears as the first term). Moreover, such sequences are determined up to isomorphism by either the first or the last term (see for reference [2]). One can do a similar construction in the context of higher dimensional Auslander–Reiten theory, at the cost of restricting to a suitable subcategory $\mathcal{C}$ of $\operatorname{mod} A$ that contains all injectives and all projectives. Then one gets longer so called n-almost split sequences

$$0 \to M \to X_1 \to \cdots \to X_n \to L \to 0$$

in $\mathcal{C}$, and again every nonprojective module in $\mathcal{C}$ appears at the end of such a sequence and every noninjective at the start of one. Again, these sequences are determined by their first or last term (see [8,9]). One of the most basic cases where such a situation appears is when A is n-representation finite (cf. [6,8]).

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Definition. Let A be a finite-dimensional k-algebra, and let $n \in \mathbb{Z}_{>0}$. An *n-cluster tilting module* for A is a module $M_A \in \operatorname{mod} A$ such that

$$\begin{aligned}\operatorname{add} M_A &= \left\{X \in \operatorname{mod} A \mid \operatorname{Ext}^i_A(M_A, X) = 0 \text{ for every } 0 < i < n\right\} = \\ &= \left\{X \in \operatorname{mod} A \mid \operatorname{Ext}^i_A(X, M_A) = 0 \text{ for every } 0 < i < n\right\}.\end{aligned}$$

We say that A is *n-representation finite* if $\operatorname{gl.dim} A \leq n$ and there exists an n-cluster tilting module for A. Then $\operatorname{gl.dim} A = 0$ or $\operatorname{gl.dim} A = n$.

For such algebras it is known that $\operatorname{add} M_A$ is a subcategory of $\operatorname{mod} A$ that admits n-almost split sequences. We call D the functor $D = \operatorname{Hom}_k(-, k) : \operatorname{mod} A \to A \operatorname{mod}$. The (higher) *Auslander–Reiten translations* τ_n, τ_n^- are defined as follows:

$$\tau_n = D \operatorname{Ext}^n_A(-, A) : \operatorname{mod} A \to \operatorname{mod} A$$

$$\tau_n^- = \operatorname{Ext}^n_A(DA, -) : \operatorname{mod} A \to \operatorname{mod} A.$$

It is immediate from this definition that

$$\tau_n A = 0 = \tau_n^- DA.$$

These higher Auslander–Reiten translations behave similarly to the classical ones.

Theorem. *Let A be an n-representation finite k-algebra. Let $P_1, \ldots, P_a$ be nonisomorphic representatives of the isomorphism classes of indecomposable projective right A-modules, and $I_1, \ldots, I_a$ the corresponding indecomposable injective modules. Then:*

(1) *There exist positive integers $l_1, \ldots, l_a$ and a permutation $\sigma \in S_a$ (the symmetric group over a elements) such that $P_i \cong \tau_n^{l_i - 1} I_{\sigma(i)}$ for every i.*
(2) *There exists a unique (up to isomorphism) basic n-cluster tilting module M_A, which is given by*

$$M_A = \bigoplus_{i=1}^{a} \bigoplus_{j=0}^{l_i - 1} \tau_n^j I_{\sigma(i)}.$$

(3) *The Auslander–Reiten translations induce mutually quasi-inverse equivalences*

$$\begin{tikzcd} \operatorname{add}(M_A/P) \arrow[r, shift right, "\tau_n^-"'] & \operatorname{add}(M_A/I) \arrow[l, shift right, "\tau_n"'] \end{tikzcd}$$

where $P = \bigoplus_{i=1}^{a} P_i$ and $I = \bigoplus_{i=1}^{a} I_i$.

Proof. See [9, 1.3(b)]. $\square$

From the last point it follows in particular that the n-cluster tilting module can be equally described by

$$M_A = \bigoplus_{i=1}^{a} \bigoplus_{j=0}^{l_i - 1} \tau_n^{-j} P_i.$$

Definition *([6]).* An n-representation finite algebra A is said to be *l-homogeneous* if with the above notation we have $l_1 = \cdots = l_a = l$.

If A is n-representation finite, the category $\operatorname{add} M_A$ decomposes into "slices", in the sense that every $X \in \operatorname{add} M_A$ can be written uniquely as $X \cong \bigoplus_{i\geq 0} X_i$, where each $X_i \in \operatorname{add} \tau_n^{-i} A$. If A is l-homogeneous, then every slice $\operatorname{add} \tau_n^{-j} A$, where $0 \leq j \leq l-1$, has the same number of isomorphism classes of indecomposables.

We denote by $\mathcal{D}^b(\operatorname{mod} A)$ the bounded derived category of $\operatorname{mod} A$, and denote by $\varepsilon : \operatorname{mod} A \to \mathcal{D}^b(\operatorname{mod} A)$ the natural inclusion. The Nakayama functors

$$\nu = - \overset{L}{\underset{A}{\otimes}} DA \cong D \circ R\operatorname{Hom}_A(-, A) : \mathcal{D}^b(\operatorname{mod} A) \to \mathcal{D}^b(\operatorname{mod} A)$$
$$\nu^{-1} = R\operatorname{Hom}_{A^{op}}(D-, A) \cong R\operatorname{Hom}_A(DA, -) : \mathcal{D}^b(\operatorname{mod} A) \to \mathcal{D}^b(\operatorname{mod} A)$$

are quasi-inverse equivalences that make the diagram

$$\begin{tikzcd} \mathcal{D}^b(\operatorname{mod} A) \arrow[r, "\nu"] & \mathcal{D}^b(\operatorname{mod} A) \\ \mathcal{K}^b(\operatorname{proj} A) \arrow[r, "\nu"] \arrow[u] & \mathcal{K}^b(\operatorname{inj} A) \arrow[u] \end{tikzcd}$$

commute ($\mathcal{K}^b$ denotes the bounded homotopy category). If A is n-representation finite, there is a natural isomorphism of functors $\operatorname{mod} A \to \operatorname{mod} A$

$$\tau_n \cong H_0 \circ \nu_n \circ \varepsilon$$

where $\nu_n = \nu \circ [-n]$. For every i and for every $0 \leq j \leq l_i$, we have that $\varepsilon \tau_n^{-j} P_i = \nu_n^{-j} \varepsilon P_i$. From now on, explicit mentions of ε will be omitted for simplicity.

The definition of higher almost split sequences that is convenient to take is the following:

Definition. Let A be an n-representation finite k-algebra, and let M_A be the corresponding basic n-cluster tilting module. Let

$$0 \longrightarrow C_{n+1} \xrightarrow{f_{n+1}} C_n \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{f_1} C_0 \longrightarrow 0$$

be an exact sequence with terms in $\operatorname{add} M_A$. Such a sequence is an *n-almost split sequence* if the following holds:

(1) For every i, we have $f_i \in \operatorname{rad}(C_i, C_{i-1})$.
(2) The modules C_{n+1} and C_0 are indecomposable.
(3) The sequence of functors from $\operatorname{add} M_A$ to $k \operatorname{mod}$

$$0 \longrightarrow \operatorname{Hom}_A(-, C_{n+1}) \xrightarrow{f_{n+1}\circ -} \operatorname{Hom}_A(-, C_n) \longrightarrow \cdots$$
$$\cdots \longrightarrow \operatorname{Hom}_A(-, C_1) \xrightarrow{f_1\circ -} \operatorname{rad}_A(-, C_0) \longrightarrow 0$$

is exact (i.e. it is an exact sequence when evaluated at any $X \in \operatorname{add} M_A$).

Theorem. *Let A be an n-representation finite k-algebra, and let M_A be the corresponding basic n-cluster tilting module. Then we have the following:*

(1) *For every indecomposable nonprojective module $N \in \operatorname{add} M_A$ there exists an n-almost split sequence*

$$0 \longrightarrow \tau_n N \longrightarrow \cdots \longrightarrow N \longrightarrow 0,$$

and any n-almost split sequence whose last term is N is isomorphic to this one.

(2) *For every indecomposable noninjective module $M \in \operatorname{add} M_A$ there exists an n-almost split sequence*

$$0 \longrightarrow M \longrightarrow \cdots \longrightarrow \tau_n^- M \longrightarrow 0,$$

and any n-almost split sequence whose first term is M is isomorphic to this one.

Proof. See [7, Theorem 3.3.1]. Notice that the term "n-cluster tilting subcategory" has replaced "$(n-1)$-orthogonal subcategory" in recent literature. $\square$

Remark. The usual, more general definition of n-almost split sequences that one takes requires that the condition dual to (3) holds as well (as in [9, Definition 2.1]). However, in the case we are considering (module categories over an n-representation finite algebra), the two definitions are equivalent (see [8, Proposition 2.10]).

In their paper [6], Herschend and Iyama construct a class of examples of n-representation finite algebras via tensor products, in the setting where the ground field k is perfect. Namely, they find a necessary and sufficient condition (being l-homogeneous for the same value of l) under which the tensor product $A \otimes B = A \otimes_k B$ of an n-representation finite algebra A with an m-representation finite algebra B is $(n+m)$-representation finite. They also show that in this case every indecomposable of $\operatorname{add} M_{A\otimes B}$ is of the form $L \otimes N$ for some indecomposables $L \in \operatorname{add} M_A$ and $N \in \operatorname{add} M_B$, and that $\tau_{n+m}^{\pm} L \otimes N \cong \tau_n^{\pm} L \otimes \tau_m^{\pm} N$. Moreover, in this case the algebra $A \otimes B$ is itself l-homogeneous.

Remark. Even though not explicitly stated in [6], necessity of the condition comes from the following observation. Let

$$M = \bigoplus_{i,j} \bigoplus_{d} \tau_{n+m}^{-d} P_i \otimes Q_j$$

where P_i and Q_j run over the indecomposable summands of A, B respectively. If A and B are not l-homogeneous for the same value of l, then M has either an indecomposable summand of the form $S = L \otimes J$ where J is injective and L is not, or one of the form $S = I \otimes N$ where I is injective and N is not. On the other hand, if $A \otimes B$ is $(n+m)$-representation finite, then M is an $(n+m)$-cluster tilting module, and hence the indecomposable injective $A \otimes B$-modules are precisely those indecomposable direct summands $I \otimes J$ of M such that $\tau_{n+m}^- I \otimes J = 0$. Thus we reach a contradiction, since S is not injective, but $\tau_{n+m}^- S = 0$.

In this setting, if

$$0 \to L \otimes N \to \cdots \to \tau_{n+m}^- L \otimes N \to 0$$

is an $(n+m)$-almost split sequence, then $\tau_{n+m}^- L \otimes N \cong \tau_n^- L \otimes \tau_m^- N$. On the other hand, there are n-respectively m-almost split sequences

$$0 \to L \to \cdots \to \tau_n^- L \to 0$$

and

$$0 \to N \to \cdots \to \tau_m^- N \to 0,$$

so the starting and ending points behave well with respect to tensor products. It is then a natural question to describe the relation between the sequence starting in $L \otimes N$ and the sequences starting in L and N. This is the question that we address, and we answer it in the setting where A is n-representation finite, l-homogeneous and B is m-representation finite, l-homogeneous.

For a precise statement, we need some more notation. For a preadditive category $\mathcal{A}$, we denote by $\mathcal{C}(\mathcal{A})$ the category of chain complexes of $\mathcal{A}$. If A is a k-algebra and $\mathcal{A}$ is a full subcategory of $\operatorname{mod} A$, we denote by $\mathcal{C}_r(\mathcal{A})$ the full subcategory of $\mathcal{C}(\mathcal{A})$ whose objects are chain complexes where the differentials are radical morphisms (i.e. $d_i \in \operatorname{rad}(A_i, A_{i-1})$ for every i). Let $\mathcal{B}$ be a full subcategory of $\mathcal{C}(\mathcal{A})$. We denote by $\operatorname{Mor}(\mathcal{B})$ the category whose objects are chain maps $A_\bullet \to B_\bullet$ for $A_\bullet, B_\bullet \in \mathcal{B}$, and whose morphisms are the obvious commutative diagrams. We denote by $\operatorname{Mor}_r(\mathcal{B})$ the full subcategory of $\operatorname{Mor}(\mathcal{B})$ whose objects are radical chain maps $A_\bullet \to B_\bullet$ for $A_\bullet, B_\bullet \in \mathcal{B}$ (meaning that for every i the map $A_i \to B_i$ is radical). We often view finite (exact) sequences as bounded chain complexes, and unless otherwise specified the degree-0 term is the rightmost nonzero term. With this point of view in mind, we denote by $\mathcal{B}^n$ the full subcategory of $\mathcal{B}$ whose objects are complexes $C_\bullet$ satisfying $C_i = 0$ for every $i < 0$ and $i > n$.

Definition. Let A be an n-representation finite k-algebra, and let $i \in \mathbb{Z}_{\geq 0}$. Let $L \in \operatorname{add} M_A$ be indecomposable noninjective, and let $C_\bullet$ be the corresponding n-almost split sequence. Then we say that $C_\bullet$ *starts in slice* i if $L \in \operatorname{add} \tau_n^{-i} A$.

We denote by Cone the mapping cone (see Definition 2.1). We use the symbol $\otimes^T$ for the usual "total tensor product" bifunctor

$$- \otimes^T - : \mathcal{C}(\operatorname{mod} A) \times \mathcal{C}(\operatorname{mod} B) \to \mathcal{C}(\operatorname{mod} A \otimes B)$$

induced by $\otimes$ (see Section 3 for details). Our main result is the following:

Theorem 1.1. *Let k be a perfect field. Let A and B be n- respectively m-representation finite k-algebras. Suppose that A and B are l-homogeneous for some common l. Let $\varphi \in \operatorname{Mor}_r(\mathcal{C}_r(\operatorname{add} M_A))$ and let $\psi \in \operatorname{Mor}_r(\mathcal{C}_r(\operatorname{add} M_B))$. Suppose that $\operatorname{Cone}(\varphi)$ and $\operatorname{Cone}(\psi)$ are n- respectively m-almost split sequences starting in slice i for some common $i \geq 0$. Then $\operatorname{Cone}(\varphi \otimes^T \psi)$ is an $(n+m)$-almost split sequence.*

Remark. In Theorem 2.4 we show that every n-almost split sequence is isomorphic to $\operatorname{Cone}(\varphi)$ for some suitable φ, so all the $(n+m)$-almost split sequences in $\operatorname{mod}(A \otimes B)$ are obtained by this procedure.

Remark. The sequence $\operatorname{Cone}(\varphi \otimes^T \psi)$ starts in slice i. This is because we have (see [6])

$$\tau_n^{-i} A \otimes \tau_m^{-i} B = \tau_{n+m}^{-i} A \otimes B.$$

On the other hand, if $L \in \operatorname{add} \tau_n^{-i} A$ and $N \in \operatorname{add} \tau_m^{-j} B$ with $i \neq j$, then $L \otimes N \notin \operatorname{add} M_{A \otimes B}$, so there is in principle no $(n+m)$-almost split sequence starting in $L \otimes N$.

Remark. If we drop the condition guaranteeing that $A \otimes B$ is $(n+m)$-representation finite, then we can perform the same construction, and we still get some sequences in $\operatorname{mod}(A \otimes B)$ which retain some interesting properties. Similarly, one could tensor sequences that do not start in the same slice. This is a possible topic for future investigation.

In Section 2 we show that every n-almost split sequence over an n-representation finite algebra is isomorphic to the mapping cone of a suitable chain map of complexes, then we relate the property of being n-almost split to a property of the chain map. In Section 3 we define the functor $\otimes^T$ that we have mentioned above, and we prove the main theorem. In Section 4 we compute an example where we explicitly construct a 2-almost split sequence and a 3-almost split sequence starting from a 1-representation finite algebra.

Conventions. Throughout this paper, we denote by k a perfect field (cf. [6]). All k-algebras are associative and unitary. For a ring R, we denote by $\operatorname{mod} R$ (resp. $R \operatorname{mod}$) the category of finitely generated right (resp. left) R-modules. Unless otherwise specified, modules are right modules. Subcategory means full subcategory. For a k-algebra A we denote by $\operatorname{rad}_A(-,-)$ the subfunctor of $\operatorname{Hom}_A(-,-)$ defined by

$$\operatorname{rad}_A(X,Y) = \{f \in \operatorname{Hom}_A(X,Y) \mid \operatorname{id}_X - g \circ f \text{ is invertible } \forall g \in \operatorname{Hom}_A(Y,X)\}$$

for all A-modules X, Y (see [2, Appendix 3]). Thus $\operatorname{rad}_A(-,-)$ is biadditive, and for two indecomposable modules $X \ncong Y$ we have $\operatorname{rad}_A(X,Y) = \operatorname{Hom}_A(X,Y)$. Moreover, for an indecomposable module X we have that $\operatorname{rad}_A(X) := \operatorname{rad}_A(X,X)$ is the Jacobson radical of the algebra $\operatorname{End}_A(X)$. We denote by $S_A(X,Y)$ the quotient $S_A(X,Y) = \operatorname{Hom}_A(X,Y)/\operatorname{rad}_A(X,Y)$ (and sometimes write only $S_A(X)$ instead of $S_A(X,X)$). To simplify the notation, we sometimes omit the reference to the algebra when this is clear from the context (writing for instance Hom instead of Hom_A). For the rest of this paper, fix finite-dimensional k-algebras A and B, where A is n-representation finite and B is m-representation finite. Set $\mathcal{A} = \operatorname{add} M_A$, $\mathcal{B} = \operatorname{add} M_B$, $\mathcal{A}_i = \operatorname{add} \tau_n^{-i} A$ for $i \geq 0$, and $\mathcal{B}_j = \operatorname{add} \tau_m^{-j} B$ for $j \geq 0$.

2. n-Almost split sequences as mapping cones

2.1. Preliminaries

If A is n-representation finite, then the morphisms in $\mathcal{A}$ are "directed" with respect to the action of τ_n^-. More precisely, we have the following:

Proposition 2.1. *Let A be an n-representation finite k-algebra. Let $M \in \mathcal{A}_i$ and $N \in \mathcal{A}_j$ with $i > j$. Then*

$$\operatorname{Hom}_A(M,N) = 0.$$

Proof. It is enough to check the result for M, N indecomposable, i.e. $M \cong \tau_n^{-i} P_1$ and $N \cong \tau_n^{-j} P_2$ for some indecomposable projectives $P_1, P_2 \in \operatorname{add} A$. We have

$$\begin{aligned}\operatorname{Hom}_A(M,N) = \operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(M,N) = \operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(\nu_n^{-i} P_1, \nu_n^{-j} P_2) = \\ = \operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(P_1, \nu_n^{i-j} P_2).\end{aligned}$$

In particular, $\operatorname{Hom}_A(M,N)$ is a direct summand of (with the previous notation)

$$\begin{aligned}\bigoplus_{i=1}^{a} \operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(P_i, \nu_n^{i-j} P_2) = \operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(A, \nu_n^{i-j} P_2) = \\ = H_0(\nu_n^{i-j} P_2) = \tau_n^{i-j} P_2 = 0\end{aligned}$$

since $i > j$ and P_2 is projective, so we are done. □

Remark 2.1. For $n = 1$, this is a special case of [1, Corollary VIII.1.4], since "1-representation finite" means "hereditary and representation finite".

We will be interested in checking whether a given complex is an n-almost split sequence, and for this purpose it is convenient to take a slightly different point of view on the definition of n-almost splitness. Namely, fix an object $X \in \mathcal{A}$. We can define a functor $F_X : \mathcal{C}_r(\mathcal{A}) \to \mathcal{C}(k\,\mathrm{mod})$ by mapping

$$C_\bullet = \cdots \xrightarrow{f_{i+1}} C_i \xrightarrow{f_i} \cdots \xrightarrow{f_1} C_0 \xrightarrow{f_0} \cdots$$

to

$$F_X(C_\bullet) = \cdots \xrightarrow{f_{i+1}\circ -} \mathrm{Hom}(X, C_i) \xrightarrow{f_i \circ -} \cdots \xrightarrow{f_1 \circ -} \mathrm{rad}(X, C_0) \xrightarrow{f_0 \circ -} \cdots$$

(that is, F_X is the subfunctor of $\mathrm{Hom}(X, -)$ given by replacing $\mathrm{Hom}(X, C_0)$ with $\mathrm{rad}(X, C_0)$). This is well defined since f_1 is a radical morphism, hence the image of $f_1 \circ -$ lies in $\mathrm{rad}(X, C_0)$. Then for a complex $C_\bullet \in \mathcal{C}_r(\mathcal{A})$ such that $C_i = 0$ for $i > n+1$ and $i < 0$, saying that it is an n-almost split sequence is equivalent to saying that C_{n+1} and C_0 are indecomposable, $C_\bullet$ is exact, and $F_X(C_\bullet)$ is exact for every $X \in \mathcal{A}$ (or equivalently, for every indecomposable $X \in \mathcal{A}$). Similarly, we can define a subfunctor G_X of the contravariant functor $\mathrm{Hom}(-, X) : \mathcal{C}_r(\mathcal{A}) \to \mathcal{C}(k\,\mathrm{mod})$ by mapping $C_\bullet$ to

$$G_X(C_\bullet) = \cdots \xrightarrow{-\circ f_0} \mathrm{Hom}(C_0, X) \xrightarrow{-\circ f_1} \cdots \xrightarrow{-\circ f_{n+1}} \mathrm{rad}(C_{n+1}, X) \xrightarrow{-\circ f_{n+2}} \cdots$$

This is again well defined, and if $C_\bullet \in \mathcal{C}_r(\mathcal{A})$ is n-almost split then $G_X(C_\bullet)$ is exact for every $X \in \mathcal{A}$ (cf. [8, Proposition 2.10]).

2.2. From sequences to cones

Definition 2.1. Let $\mathcal{D}$ be an abelian category. Let $A_\bullet \in \mathcal{C}(\mathcal{D})$ with differentials $d_i : A_i \to A_{i-1}$. For any $m \in \mathbb{Z}$, the *shift* $A[m]_\bullet$ of $A_\bullet$ is the complex with objects $A[m]_i = A_{i+m}$ and differentials $d[m]_i : A[m]_i \to A[m]_{i-1}$ given by $d[m]_i = (-1)^m d_{i+m}$ for every i.

Let $(A_\bullet, d^A_\bullet)$ and $(B_\bullet, d^B_\bullet)$ be complexes in $\mathcal{C}(\mathcal{A})$. Let $f : A_\bullet \to B_\bullet$ be a morphism of complexes with components $f_i : A_i \to B_i$. The *shift* of f is the morphism $f[m] : A_\bullet[m] \to B_\bullet[m]$ with components $f[m]_i = f_{i+m}$. Thus $[m]$ is an endofunctor on $\mathcal{C}(\mathcal{D})$. The *mapping cone* $\mathrm{Cone}(f)$ of f is the complex with objects

$$\mathrm{Cone}(f)_i = A[-1]_i \oplus B_i$$

and differentials

$$d_i^{\mathrm{Cone}(f)} = \begin{bmatrix} d[-1]^A_i & 0 \\ f[-1]_i & d^B_i \end{bmatrix}.$$

Lemma 2.2. *Let $\mathcal{D}$ be an abelian category, and let f be a morphism of complexes in $\mathcal{C}(\mathcal{D})$. Then $\mathrm{Cone}(f)$ is exact if and only if f is a quasi-isomorphism.*

Proof. This follows straight from [5, III.18]. □

Let A be n-representation finite, and let

$$C_\bullet = 0 \longrightarrow C_{n+1} \xrightarrow{f_{n+1}} C_n \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{f_1} C_0 \longrightarrow 0$$

be an n-almost split sequence starting in slice i_0 for some $i_0 \in \mathbb{Z}_{\geq 0}$. Then we can decompose the modules appearing in the sequence according to the slice decomposition of $\mathcal{A}$, i.e. we write

$$C_m = \bigoplus_{j \geq 0} B_m^j$$

with $B_m^j \in \mathcal{A}_j$ for every m, j. We know that $C_{n+1} \in \mathcal{A}_{i_0}$ and $C_0 \in \mathcal{A}_{i_0+1}$ are indecomposable. A first result, which can be seen as a generalisation of [1, Lemma VIII.1.8(b)], is the following:

Lemma 2.3. *With the above notation, we have*

$$B_m^j = 0 \textit{ for any } m \textit{ and for } j \notin \{i_0, i_0 + 1\}.$$

Proof. To reach a contradiction, suppose that the claim is false. Then there is $B_q^j \neq 0$ with $j \notin \{i_0, i_0 + 1\}$. Suppose $j > i_0 + 1$, and pick j maximal such. We can assume q minimal for that value of j, i.e. $B_{q-p}^j = 0$ for all $p > 0$. Notice that since $C_0 = B_0^{i_0+1}$ it follows that $q > 0$. We want to prove that $C_\bullet$ cannot be n-almost split in this case, and it is enough to show that $F_{B_q^j}(C_\bullet)$ is not exact. By Proposition 2.1,

$$\operatorname{Hom}(B_{p'}^j, B_p^i) = 0$$

for every p, p' and for every $i < j$. By maximality of j, we get that $B_\bullet^j$ is a subcomplex of $C_\bullet$, and

$$F_{B_q^j}(C_\bullet) = F_{B_q^j}(B_\bullet^j).$$

Since q is minimal and $q > 0$ we can write explicitly

$$F_{B_q^j}(C_\bullet) = \ \cdots \longrightarrow \operatorname{Hom}(B_q^j, B_m^j) \longrightarrow \cdots \xrightarrow{d} \operatorname{Hom}(B_q^j, B_q^j) \longrightarrow 0.$$

The map d in this sequence is composition with a radical morphism, so in particular it cannot be surjective on $\operatorname{Hom}(B_q^j, B_q^j)$. The sequence is then not exact and we have proved that $B_m^j = 0$ for $j > i_0 + 1$.

Suppose now that $j < i_0$, and pick j minimal such. We can assume that q is maximal for that j, i.e. $B_{q+p}^j = 0$ for all $p > 0$. Notice that since $C_{n+1} = B_{n+1}^{i_0}$ it follows that $q < n + 1$. We prove that $C_\bullet$ is not n-almost split in this case by showing that $G_{B_q^j}(C_\bullet)$ is not exact. Again by Proposition 2.1 we know that

$$\operatorname{Hom}(B_p^i, B_{p'}^j) = 0$$

for all p, p' if $i > j$. Then by minimality of j and maximality of q we get

$$G_{B_q^j}(C_\bullet) = \ \cdots \longrightarrow \operatorname{Hom}(B_m^j, B_q^j) \longrightarrow \cdots \xrightarrow{d'} \operatorname{Hom}(B_q^j, B_q^j) \longrightarrow 0$$

and d' cannot be surjective, contradiction. Hence we have proved that $B_m^j = 0$ for $j < i_0$, which completes the proof. □

Theorem 2.4. *Let A be an n-representation finite k-algebra, and let $i_0 \in \mathbb{Z}_{\geq 0}$. Let $C_{n+1} \in \mathcal{A}_{i_0}$ be indecomposable noninjective, and let*

$$C_\bullet = \ 0 \longrightarrow C_{n+1} \xrightarrow{f_{n+1}} C_n \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{f_1} C_0 \longrightarrow 0$$

be the corresponding n-almost split sequence. Then there are complexes $A^0_\bullet \in \mathcal{C}_r(\mathcal{A}_{i_0})$, $A^1_\bullet \in \mathcal{C}_r(\mathcal{A}_{i_0+1})$, and a radical morphism of complexes $\varphi : A^0_\bullet \to A^1_\bullet$, such that $C_\bullet \cong \mathrm{Cone}(\varphi)$ in $\mathcal{C}(\mathcal{A})$.

Proof. By Lemma 2.3 we can rewrite the complex $C_\bullet$ as

$$C_m = B^{i_0}_m \oplus B^{i_0+1}_m$$

where $B^{i_0}_m \in \mathcal{A}_{i_0}$ and $B^{i_0+1}_m \in \mathcal{A}_{i_0+1}$ for every m. Moreover,

$$f_m = \begin{bmatrix} b^{i_0}_m & \xi_m \\ \gamma_m & b^{i_0+1}_m \end{bmatrix} : C_m \to C_{m-1}$$

has components $b^{i_0}_m : B^{i_0}_m \to B^{i_0}_{m-1}$, $\xi_m : B^{i_0+1}_m \to B^{i_0}_{m-1}$, $\gamma_m : B^{i_0}_m \to B^{i_0+1}_{m-1}$, and $b^{i_0+1}_m : B^{i_0+1}_m \to B^{i_0+1}_{m-1}$. Notice that by Proposition 2.1 it follows that $\xi_m = 0$ for all m. Define $A^0_m = B^{i_0}_{m+1}$, $d^{A^0}_m = -b^{i_0}_{m+1}$, $A^1_m = B^{i_0+1}_m$, $d^{A^1}_m = b^{i_0+1}_m$ and $\varphi_m = -\gamma_{m+1} : A^0_m \to A^1_m$. Then $\varphi : A^0_\bullet \to A^1_\bullet$ is a chain map since

$$d^{A^1}_m \varphi_m = -b^{i_0+1}_m \gamma_{m+1} = \gamma_m b^{i_0}_{m+1} = \varphi_{m-1} d^{A^0}_m$$

where the equality

$$b^{i_0+1}_m \gamma_{m+1} + \gamma_m b^{i_0}_{m+1} = 0$$

comes from the fact that $C_\bullet$ is a complex. Moreover, $C_\bullet \cong \mathrm{Cone}(\varphi)$ and we are done. □

Remark 2.2. In [9, Proposition 3.23] Iyama constructed certain n-almost split sequences as mapping cones of chain maps. Our Theorem 2.4 states that in the n-representation finite case, every n-almost split sequence can in fact be realised as a mapping cone.

Given that n-almost split sequences are determined up to isomorphism by their endpoints, it is interesting to address the issue of uniqueness of the map φ. Since we are not going to need it in what follows, we do not investigate this in detail. We present however a result:

Proposition 2.5. *Let A be an n-representation finite algebra. Let $A^0_\bullet, B^0_\bullet \in \mathcal{C}(\mathcal{A}_{i_0})$, $A^1_\bullet, B^1_\bullet \in \mathcal{C}(\mathcal{A}_{i_0+1})$. Let $\varphi : A^0_\bullet \to A^1_\bullet$ and $\psi : B^0_\bullet \to B^1_\bullet$ be chain maps. Then the following are equivalent:*

(1) $\mathrm{Cone}(\varphi) \cong \mathrm{Cone}(\psi)$ *in $\mathcal{C}(\mathcal{A})$.*
(2) *There are isomorphisms of complexes $f : A^0_\bullet \to B^0_\bullet, g : A^1_\bullet \to B^1_\bullet$ such that the diagram*

$$\begin{tikzcd} A^0_\bullet \arrow[r, "f"] \arrow[d, "\varphi"'] & B^0_\bullet \arrow[d, "\psi"] \\ A^1_\bullet \arrow[r, "g"] & B^1_\bullet \end{tikzcd}$$

commutes in the homotopy category $\mathcal{K}(\mathcal{A})$.

Proof. Let us begin by some observations. Let

$$\alpha_m = \begin{bmatrix} a_m & r_m \\ q_m & b_m \end{bmatrix} : A^0_{m-1} \oplus A^1_m \to B^0_{m-1} \oplus B^1_m$$

be a morphism of modules. Notice that by Proposition 2.1, we have $r_m = 0$. Observe now that

$$\begin{aligned}
&(\alpha_m) \text{ defines a chain map } \alpha : \mathrm{Cone}(\varphi) \to \mathrm{Cone}(\psi) \\
\Leftrightarrow &\begin{bmatrix} a_{m-1} & 0 \\ q_{m-1} & b_{m-1} \end{bmatrix} \begin{bmatrix} -d^{A^0}_{m-1} & 0 \\ \varphi_{m-1} & d^{A^1}_m \end{bmatrix} = \begin{bmatrix} -d^{B^0}_{m-1} & 0 \\ \psi_{m-1} & d^{B^1}_m \end{bmatrix} \begin{bmatrix} a_m & 0 \\ q_m & b_m \end{bmatrix} \quad \text{for all } m \\
\Leftrightarrow &\begin{cases} a_{m-1} d^{A^0}_{m-1} = d^{B^0}_{m-1} a_m \quad \text{for all } m \\ b_{m-1} d^{A^1}_m = d^1_m b_m \quad \text{for all } m \\ b_{m-1}\varphi_{m-1} = \psi_{m-1} a_m + d^{B^1}_m q_m + q_{m-1} d^{A^0}_{m-1} \quad \text{for all } m \end{cases} \\
\Leftrightarrow &\begin{cases} (a_m) \text{ defines a chain map } a : A^0[-1]_\bullet \to B^0[-1]_\bullet \\ (b_m) \text{ defines a chain map } b : A^1_\bullet \to B^1_\bullet \\ b_{m-1}\varphi_{m-1} = \psi_{m-1} a_m + d^{B^1}_m q_m + q_{m-1} d^{A^0}_{m-1} \quad \text{for all } m. \end{cases}
\end{aligned}$$

Now let us prove (1) $\Rightarrow$ (2). Use the same notation as above, and assume that α is an isomorphism. That means that α_m is an isomorphism for every m. Since $A^0_{m-1} \in \mathcal{A}_{i_0}$ and $B^1_m \in \mathcal{A}_{i_0+1}$, it follows that no indecomposable direct summand of A^0_{m-1} can be isomorphic to a direct summand of B^1_m, hence $\mathrm{Hom}(A^0_{m-1}, B^1_m) = \mathrm{rad}(A^0_{m-1}, B^1_m)$. In particular we have that q_m is a radical map. Since α_m has an inverse, both a_m and b_m have inverses modulo radical morphisms. This means that there are $x : B^0_{m-1} \to A^0_{m-1}, y : B^1_m \to A^1_m$ such that

$$\begin{gathered}
a_m x - \mathrm{id}_{B^0_{m-1}}, \\
x a_m - \mathrm{id}_{A^0_{m-1}}, \\
b_m y - \mathrm{id}_{B^1_m}, \\
y b_m - \mathrm{id}_{B^1_m}
\end{gathered}$$

are radical morphisms. In particular $a_m x, x a_m, b_m y, y b_m$ are all invertible, hence a_m and b_m are isomorphisms. By the above observations, $a[1]$ and b are well-defined isomorphisms of complexes, and since

$$\left(d^{B^1}_m q_m + q_{m-1} d^{A^0}_{m-1}\right) : A^0_\bullet \to B^1_\bullet$$

is null-homotopic we obtain that the diagram

$$\begin{tikzcd}
A^0_\bullet \arrow[r, "{a[1]}"] \arrow[d, "\varphi"'] & B^0_\bullet \arrow[d, "\psi"] \\
A^1_\bullet \arrow[r, "b"] & B^1_\bullet
\end{tikzcd}$$

commutes in $\mathcal{K}(\mathcal{A})$ as required.

Let us now prove (2) $\Rightarrow$ (1). Since the diagram commutes in $\mathcal{K}(\mathcal{A})$, there is a homotopy $\left(q_m : A^0_{m-1} \to B^1_m\right)$ such that

$$b_{m-1}\varphi_{m-1} = \psi_{m-1} a_m + d^{B^1}_m q_m + q_{m-1} d^{A^0}_{m-1} \quad \text{for all } m.$$

By the above observations, setting for every m

$$\alpha_m = \begin{bmatrix} f_{m-1} & 0 \\ q_m & g_m \end{bmatrix} : A^0_{m-1} \oplus A^1_m \to B^0_{m-1} \oplus B^1_m$$

defines a chain map $\alpha : \mathrm{Cone}(\varphi) \to \mathrm{Cone}(\psi)$. It remains to check that α is an isomorphism, which amounts to checking that α_m is invertible for all m. Since we are assuming that f and g are isomorphisms, we can define for every m

$$\beta_m = \begin{bmatrix} f^{-1}_{m-1} & 0 \\ -g^{-1}_m q_m f^{-1}_{m-1} & g^{-1}_m \end{bmatrix} : B^0_{m-1} \oplus B^1_m \to A^0_{m-1} \oplus A^1_m.$$

It is then a straightforward computation to check that β_m is the inverse of α_m, and we are done. □

2.3. From cones to sequences

Since we can realise any n-almost split sequence as $\mathrm{Cone}(\varphi)$ for some φ, it makes sense to relate the property of being n-almost split to the properties of φ. Let us introduce some more notation. For a given $X \in \mathcal{A}$, we define a functor $\tilde{F}_X : \mathrm{Mor}_r(\mathcal{C}_r(\mathcal{A})) \to \mathrm{Mor}(\mathcal{C}(k\,\mathrm{mod}))$ by mapping $\varphi : A_\bullet \to B_\bullet$ to

$$\tilde{F}_X(\varphi) = \varphi \circ - : \mathrm{Hom}(X, A_\bullet) \to F_X(B_\bullet),$$

where $\mathrm{Hom}(X, A_\bullet)$ denotes the complex $\cdots \to \mathrm{Hom}(X, A_i) \to \mathrm{Hom}(X, A_{i-1}) \to \cdots$. This is well defined because $\varphi_0 \in \mathrm{rad}(A_0, B_0)$.

Consider the mapping cone functor $\mathrm{Cone} : \mathrm{Mor}_r(\mathcal{C}_r(\mathcal{A})) \to \mathcal{C}(\mathcal{A})$. By definition, this factors through the inclusion $\mathcal{C}_r(\mathcal{A}) \to \mathcal{C}(\mathcal{A})$, and we still denote by Cone the corresponding functor $\mathrm{Cone} : \mathrm{Mor}_r(\mathcal{C}_r(\mathcal{A})) \to \mathcal{C}_r(\mathcal{A})$. We also denote by Cone the mapping cone functor $\mathrm{Cone} : \mathrm{Mor}(\mathcal{C}(k\,\mathrm{mod})) \to \mathcal{C}(k\,\mathrm{mod})$.

Lemma 2.6. *With the above notation, we have that the diagram*

$$\begin{tikzcd} \mathrm{Mor}_r(\mathcal{C}^n_r(\mathcal{A})) \arrow[r, "\tilde{F}_X"] \arrow[d, "\mathrm{Cone}"'] & \mathrm{Mor}(\mathcal{C}^n(k\,\mathrm{mod})) \arrow[d, "\mathrm{Cone}"] \\ \mathcal{C}_r(\mathcal{A}) \arrow[r, "F_X"] & \mathcal{C}(k\,\mathrm{mod}) \end{tikzcd}$$

commutes for every $X \in \mathcal{A}$ and for any choice of $n \in \mathbb{Z}_{\geq 0}$.

Proof. Pick a morphism $\varphi : A_\bullet \to B_\bullet \in \mathrm{Mor}_r(\mathcal{C}^n_r(\mathcal{A}))$. Then

$$\begin{aligned} \mathrm{Cone}(\tilde{F}_X(\varphi))_i &= \mathrm{Hom}(X, A_{i-1}) \oplus F_X(B_i) = \\ &= \begin{cases} \mathrm{Hom}(X, A_{i-1}) \oplus \mathrm{Hom}(X, B_i) & \text{if } \; i \neq 0 \\ \mathrm{Hom}(X, A_{-1}) \oplus \mathrm{rad}(X, B_0) = \mathrm{rad}(X, B_0) & \text{if } \; i = 0 \end{cases} \end{aligned}$$

and the differential $d_i : \mathrm{Cone}(\tilde{F}_X(\varphi))_i \to \mathrm{Cone}(\tilde{F}_X(\varphi))_{i-1}$ is given by

$$d_i = \begin{bmatrix} -d^A_{i-1} \circ - & 0 \\ -\varphi_{i-1} \circ - & d^B_i \circ - \end{bmatrix}.$$

On the other hand, we have

$$F_X(\mathrm{Cone}(\varphi))_i = \begin{cases} \mathrm{Hom}(X, A_{i-1} \oplus B_i) = \mathrm{Hom}(X, A_{i-1}) \oplus \mathrm{Hom}(X, B_i) & \text{if } \; i \neq 0 \\ \mathrm{rad}(X, A_{-1} \oplus B_0) = \mathrm{rad}(X, B_0) & \text{if } \; i = 0 \end{cases}$$

and the differential $d'_i : F_X(\mathrm{Cone}(\varphi))_i \to F_X(\mathrm{Cone}(\varphi))_{i-1}$ is given by

$$d'_i = d_i^{\mathrm{Cone}(\varphi)} \circ - = \begin{bmatrix} -d^A_{i-1} \circ - & 0 \\ -\varphi_{i-1} \circ - & d^B_i \circ - \end{bmatrix}. \qquad \square$$

We get a useful criterion for checking whether the cone of a chain map is an n-almost split sequence.

Lemma 2.7 *(Criterion for n-almost splitness). Let $A^0_\bullet \in \mathcal{C}^n_r(\mathcal{A}_{i_0})$, $A^1_\bullet \in \mathcal{C}^n_r(\mathcal{A}_{i_0+1})$ for some i_0. Let $\varphi : A^0_\bullet \to A^1_\bullet$ be a chain map. Then the following are equivalent:*

(1) $\mathrm{Cone}(\varphi)$ *is an n-almost split sequence.*
(2) A^0_n *and* A^1_0 *are indecomposable, and $\tilde{F}_X(\varphi)$ is a quasi-isomorphism for every $X \in \mathcal{A}$.*

Proof. (1) $\Rightarrow$ (2). Suppose that $\mathrm{Cone}(\varphi)$ is n-almost split. Then by definition $A^0_n = \mathrm{Cone}(\varphi)_{n+1}$ and $A^1_0 = \mathrm{Cone}(\varphi)_0$ are indecomposable and $F_X(\mathrm{Cone}(\varphi))$ is exact for every $X \in \mathcal{A}$. By Lemma 2.6 we know that $F_X(\mathrm{Cone}(\varphi)) = \mathrm{Cone}(\tilde{F}_X(\varphi))$, and by Lemma 2.2 exactness of $\mathrm{Cone}(\tilde{F}_X(\varphi))$ implies that $\tilde{F}_X(\varphi)$ is a quasi-isomorphism.

(2) $\Rightarrow$ (1). If $\tilde{F}_X(\varphi)$ is a quasi-isomorphism for every $X \in \mathcal{A}$, then by Lemma 2.2 we know that $\mathrm{Cone}(\tilde{F}_X(\varphi))$ is exact, so by Lemma 2.6 we get that $F_X(\mathrm{Cone}(\varphi))$ is exact for every $X \in \mathcal{A}$. Then by observing that $\mathrm{Cone}(\varphi)_{n+1} = A^0_n$ and $\mathrm{Cone}(\varphi)_0 = A^1_0$ are indecomposable, we can conclude that $\mathrm{Cone}(\varphi)$ is n-almost split. $\square$

3. Tensor product of mapping cones

3.1. Construction

All tensor products are understood to be over k, even when it is not specified to simplify the notation. The tensor product bifunctor

$$- \otimes - : \mathrm{mod}\, k \times \mathrm{mod}\, k \to \mathrm{mod}\, k$$

induces (for a general construction, see [4, IV.4,5]) a bifunctor

$$- \otimes^T - : \mathcal{C}(\mathrm{mod}\, k) \times \mathcal{C}(\mathrm{mod}\, k) \to \mathcal{C}(\mathrm{mod}\, k)$$

(for clarity, we use the symbol $\otimes$ for modules and $\otimes^T$ for complexes). Moreover, since the tensor product defines a bifunctor

$$- \otimes - : \mathrm{mod}\, A \times \mathrm{mod}\, B \to \mathrm{mod}(A \otimes B)$$

we can consider $\otimes^T$ as a bifunctor

$$- \otimes^T - : \mathcal{C}(\mathrm{mod}\, A) \times \mathcal{C}(\mathrm{mod}\, B) \to \mathcal{C}(\mathrm{mod}\, A \otimes B).$$

For convenience, we give the explicit formulas: on objects, we have

$$(A \otimes^T B)_m = \bigoplus_{j \in \mathbb{Z}} A_j \otimes B_{m-j}$$

with differential d given on an element $v \otimes w \in A_j \otimes B_{m-j}$ by

$$d_m(v \otimes w) = d_j^A(v) \otimes w + (-1)^j v \otimes d_{m-j}^B(w).$$

On morphisms, if $\varphi : A_\bullet^0 \to A_\bullet^1$ and $\psi : B_\bullet^0 \to B_\bullet^1$ are chain maps, then

$$(\varphi \otimes^T \psi)_m = \bigoplus_{j\in\mathbb{Z}} \varphi_j \otimes \psi_{m-j} : \bigoplus_{j\in\mathbb{Z}} A_j^0 \otimes B_{m-j}^0 \to \bigoplus_{j\in\mathbb{Z}} A_j^1 \otimes B_{m-j}^1.$$

Lemma 3.1. *Let A, B be finite-dimensional k-algebras, let $\mathcal{A}, \mathcal{B}$ be subcategories of* $\operatorname{mod} A$ *and* $\operatorname{mod} B$ *respectively, and let $\varphi : A_\bullet^0 \to A_\bullet^1$ and $\psi : B_\bullet^0 \to B_\bullet^1$ be objects of* $\operatorname{Mor}(\mathcal{C}(\mathcal{A}))$ *and* $\operatorname{Mor}(\mathcal{C}(\mathcal{B}))$ *respectively. Suppose that both φ and ψ are quasi-isomorphisms. Then $\varphi \otimes^T \psi$ is a quasi-isomorphism.*

Proof. This follows from the Künneth formula over a field (see [4, VI.3.3.1]). That is, for complexes $A_\bullet$ and $B_\bullet$ there is for every n a functorial isomorphism

$$H_n(A_\bullet \otimes^T B_\bullet) \cong \bigoplus_{i+j=n} H_i(A_\bullet) \otimes H_j(B_\bullet).$$

In our case, this gives for every n an isomorphism

$$H_n(\varphi \otimes^T \psi) \cong \left(H_i(\varphi) \otimes H_j(\psi)\right)_{i+j=n}.$$

Since φ and ψ are quasi-isomorphisms, the right-hand side is an isomorphism, hence $\varphi \otimes^T \psi$ is a quasi-isomorphism. $\square$

3.2. Preparation

We now focus on the tensor product of homogeneous algebras. In this case the tensor product behaves well (recall that we are assuming k to be perfect). More precisely, we have the following classical result:

Proposition 3.2. *Let A, B be finite-dimensional k-algebras. Then*

$$\operatorname{gl.dim}(A \otimes_k B) = \operatorname{gl.dim}(A) + \operatorname{gl.dim}(B).$$

Proof. Using a result by Auslander ([3, Theorem 16]), we can assume that A and B are semisimple. Then the claim is a special case of [10, Corollary 5.7]. $\square$

In our setting, perfectness of the ground field and homogeneity are enough to guarantee that higher representation finiteness is preserved by tensor products:

Theorem 3.3. *Let A be an n-representation finite k-algebra, and let B be an m-representation finite k-algebra. If A and B are l-homogeneous, then the algebra $A \otimes_k B$ is $(n+m)$-representation finite, l-homogeneous. Moreover, an $(n+m)$-cluster tilting module for $A \otimes_k B$ is given by*

$$M_{A\otimes B} = \bigoplus_{i=0}^{l-1} \tau_n^{-i} A \otimes \tau_m^{-i} B.$$

Proof. See [6, 1.5]. $\square$

Proposition 3.4. *Let A and B be two finite-dimensional k-algebras. Let $M, N \in \operatorname{mod} A$ and $M', N' \in \operatorname{mod} B$. Then the canonical map*

$$\mathrm{Hom}_A(M,N)\otimes_k \mathrm{Hom}_B(M',N') \to \mathrm{Hom}_{A\otimes_k B}(M\otimes_k M', N\otimes_k N')$$

given by $f\otimes g \mapsto f\otimes g$ is an isomorphism of k-vector spaces.

Proof. See Proposition XI.1.2.3 and Theorem XI.3.1 in [4]. □

We will use the above identification quite freely from now on. We need two more lemmas:

Lemma 3.5. *Let R and S be finite-dimensional k-algebras. Then we have*

$$\mathrm{rad}(R)\otimes_k S + R\otimes_k \mathrm{rad}(S) = \mathrm{rad}(R\otimes_k S)$$

as ideals of $R\otimes_k S$.

Proof. This is [10, Corollary 5.8], combined with the observation that for finite-dimensional algebras the Baer radical and the Jacobson radical coincide (see [11, Proposition 10.27]). □

Lemma 3.6. *Let A and B be two finite-dimensional k-algebras. Let $M,N\in \operatorname{mod} A$ and $M',N'\in \operatorname{mod} B$. Then we have*

$$\mathrm{rad}(M,N)\otimes \mathrm{Hom}(M',N') + \mathrm{Hom}(M,N)\otimes \mathrm{rad}(M',N') = \mathrm{rad}(M\otimes M', N\otimes N')$$

as subspaces of $\mathrm{Hom}(M\otimes M', N\otimes N')$. Moreover, there is an exact sequence

$$0 \longrightarrow \mathrm{rad}(M)\otimes \mathrm{rad}(M') \xrightarrow{\begin{bmatrix}\alpha\\ -\alpha\end{bmatrix}} \begin{matrix}\mathrm{rad}(M)\otimes\mathrm{End}(M')\\ \oplus\\ \mathrm{End}(M)\otimes\mathrm{rad}(M')\end{matrix} \xrightarrow{\begin{bmatrix}\alpha & \alpha\end{bmatrix}} \mathrm{rad}(M\otimes M') \longrightarrow 0$$

where

$$\alpha : f\otimes g \mapsto f\otimes g.$$

Proof. Let $R=\mathrm{End}_A(M\oplus N)$ and $S=\mathrm{End}_B(M'\oplus N')$. By Proposition 3.4 we have

$$R\otimes S \cong \mathrm{End}_{A\otimes B}((M\oplus N)\otimes(M'\oplus N')).$$

Let $p,q\in R$ be the projections onto M,N respectively, and let $p',q'\in S$ be the projections onto M',N' respectively. Then we have

$$(q\otimes q')(\mathrm{rad}(R\otimes S))(p\otimes p') = \mathrm{rad}(M\otimes M', N\otimes N').$$

By Lemma 3.5,

$$\mathrm{rad}(R\otimes S) = \mathrm{rad}(R)\otimes S + R\otimes \mathrm{rad}(S)$$

so that

$$\begin{aligned}\mathrm{rad}(M\otimes M', N\otimes N') &= (q\otimes q')(\mathrm{rad}(R)\otimes S + R\otimes\mathrm{rad}(S))(p\otimes p') =\\ &= \mathrm{rad}(M,N)\otimes\mathrm{Hom}(M',N') + \mathrm{Hom}(M,N)\otimes\mathrm{rad}(M',N'),\end{aligned}$$

which proves the first claim. Moreover, in the case $M = N, M' = N'$ we easily get the exact sequence by looking at the kernel of the map

$$[\alpha \quad \alpha] \quad : \quad \begin{array}{c} \scriptstyle \mathrm{rad}(M)\otimes\mathrm{End}(M') \\ \oplus \\ \scriptstyle \mathrm{End}(M)\otimes\mathrm{rad}(M') \end{array} \longrightarrow \mathrm{rad}(M \otimes M'). \qquad \square$$

3.3. Proof of main result

We are ready to prove Theorem 1.1:

Proof of Theorem 1.1. We fix $\varphi : A^0_\bullet \to A^1_\bullet$ and $\psi : B^0_\bullet \to B^1_\bullet$. By definition $C_\bullet = \mathrm{Cone}(\varphi \otimes^T \psi)$ is a complex bounded between 0 and $n + m + 1$, and it is exact by Lemma 2.2 and Lemma 3.1. It follows from Lemma 3.6 that

$$(\varphi \otimes^T \psi)_i \in \mathrm{rad}((A^0_\bullet \otimes^T B^0_\bullet)_i, (A^1_\bullet \otimes^T B^1_\bullet)_i)$$

for every i, and so $C_\bullet \in \mathcal{C}_r(\mathcal{A} \otimes \mathcal{B})$. Fix an indecomposable $M \otimes N \in \mathcal{A} \otimes \mathcal{B}$. We can consider the maps

$$\tilde{F}_M(\varphi) \otimes^T \tilde{F}_N(\psi) : \mathrm{Hom}(M, A^0_\bullet) \otimes^T \mathrm{Hom}(N, B^0_\bullet) \to F_M(A^1_\bullet) \otimes^T F_N(B^1_\bullet)$$

and

$$\tilde{F}_{M\otimes N}(\varphi \otimes^T \psi) : \mathrm{Hom}(M \otimes N, A^0_\bullet \otimes^T B^0_\bullet) \to F_{M\otimes N}(A^1_\bullet \otimes^T B^1_\bullet).$$

By Lemma 3.6, the map

$$\iota : \mathrm{Hom}(M, A^1_\bullet) \otimes^T \mathrm{Hom}(N, B^1_\bullet) \to \mathrm{Hom}(M \otimes N, A^1_\bullet \otimes^T B^1_\bullet), \quad f \otimes g \mapsto f \otimes g$$

induces a monomorphism

$$\iota' : F_M(A^1_\bullet) \otimes^T F_N(B^1_\bullet) \to F_{M\otimes N}(A^1_\bullet \otimes^T B^1_\bullet)$$

so there is a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(M, A^0_\bullet) \otimes^T \mathrm{Hom}(N, B^0_\bullet) \arrow[r, "\iota"] \arrow[d, "\tilde{F}_M(\varphi) \otimes^T \tilde{F}_N(\psi)"'] & \mathrm{Hom}(M \otimes N, A^0_\bullet \otimes^T B^0_\bullet) \arrow[d, "\tilde{F}_{M\otimes N}(\varphi \otimes^T \psi)"] \\ F_M(A^1_\bullet) \otimes^T F_N(B^1_\bullet) \arrow[r, "\iota'"'] & F_{M\otimes N}(A^1_\bullet \otimes^T B^1_\bullet). \end{tikzcd}$$

By Proposition 3.4, the map ι is an isomorphism. Moreover, since $\mathrm{Cone}(\varphi)$ is n-almost split, it follows by Lemma 2.7 that $\tilde{F}_M(\varphi)$ is a quasi-isomorphism, and similarly $\tilde{F}_N(\psi)$ is a quasi-isomorphism because $\mathrm{Cone}(\psi)$ is m-almost split. Then by Lemma 3.1 it follows that $\tilde{F}_M(\varphi) \otimes^T \tilde{F}_N(\psi)$ is a quasi-isomorphism. Again by Lemma 2.7, the claim that $C_\bullet$ is $(m+n)$-almost split will follow if we prove that $\tilde{F}_{M\otimes N}(\varphi \otimes^T \psi)$ is a quasi-isomorphism (since $M \otimes N$ is an arbitrary indecomposable). By the above observations, it is enough to show that ι' is a quasi-isomorphism. This is in turn equivalent to $\mathrm{coker}\,\iota'$ being exact, which is what we prove. We claim that we have

$$\operatorname{coker}\iota' = F_M(A^1_\bullet) \otimes_k S(N, B^1_0) \oplus S(M, A^1_0) \otimes_k F_N(B^1_\bullet). \tag{1}$$

Assume that this claim holds, and let us prove the theorem. Notice that $S(N, B^1_0) = 0$ unless $N \cong B^1_0$ since B^1_0 and N are indecomposable. Suppose that $N \cong B^1_0$. Then in particular $N \in \operatorname{add} \tau_m^{-(i+1)} B$ and so $M \in \operatorname{add} \tau_n^{-(i+1)} A$ since $M \otimes N \in \mathcal{A} \otimes \mathcal{B}$ (see Theorem 3.3). Then by Proposition 2.1 we get

$$\operatorname{Hom}(M, A^0_\bullet) = 0.$$

In this case $F_M(A^1_\bullet) \cong \operatorname{Cone}(\tilde{F}_M(\varphi))$ which by Lemma 2.7 is exact if and only if $\operatorname{Cone}(\varphi)$ is n-almost split, which we are assuming. Tensoring over k is exact, so it follows that the first summand in (1) is exact. By symmetry, the second summand is exact as well and we are done.

It remains to prove the equality (1). Call $D_\bullet = F_M(A^1_\bullet) \otimes^T F_N(B^1_\bullet)$. We have that

$$D_p = \bigoplus_{i+j=p} F_M(A^1_\bullet)_i \otimes F_N(B^1_\bullet)_j$$

and we are interested in computing the cokernels of the maps

$$\iota'_p : D_p \to F_{M\otimes N}\left(A^1_\bullet \otimes^T B^1_\bullet\right)_p.$$

We proceed by first considering the case $p \neq 0$. Then the codomain of ι'_p is

$$\operatorname{Hom}\left(M \otimes N, \bigoplus_{i+j=p} A^1_i \otimes B^1_j\right) \cong \bigoplus_{i+j=p} \operatorname{Hom}(M, A^1_i) \otimes \operatorname{Hom}(N, B^1_j)$$

and ι'_p is just the canonical diagonal immersion with components

$$\iota'_{ij} : F_M(A^1_\bullet)_i \otimes F_N(B^1_\bullet)_j \to \operatorname{Hom}(M, A^1_i) \otimes \operatorname{Hom}(N, B^1_j)$$

given by $f \otimes g \mapsto f \otimes g$. In particular, ι'_{ij} is the identity unless either $i = 0$ and $M \cong A^1_0$ or $j = 0$ and $N \cong B^1_0$. It follows that

$$\operatorname{coker}\iota'_p = \bigoplus_{i+j=p} \operatorname{coker}\iota'_{ij} = \operatorname{coker}\iota'_{0p} \oplus \operatorname{coker}\iota'_{p0}.$$

Let us then suppose $N \cong B^1_0$, and focus on terms of the form $\operatorname{coker}\iota'_{p0}$, where

$$\iota'_{p0} : \operatorname{Hom}(M, A^1_p) \otimes \operatorname{rad}(B^1_0) \to \operatorname{Hom}(M \otimes B^1_0, A^1_p \otimes B^1_0).$$

We know by Proposition 3.4 that the right-hand side is canonically isomorphic to $\operatorname{Hom}(M, A^1_p) \otimes \operatorname{End}(B^1_0)$, so from the exact sequence

$$0 \longrightarrow \operatorname{rad}(B^1_0) \longrightarrow \operatorname{End}(B^1_0) \longrightarrow S(B^1_0) \longrightarrow 0$$

we conclude that $\operatorname{coker}\iota'_{p0} = \operatorname{Hom}(M, A^1_p) \otimes S(B^1_0)$. By symmetry we conclude that if $p \neq 0$ then

$$\operatorname{coker}\iota'_p = \operatorname{Hom}(M, A^1_p) \otimes S(N, B^1_0) \oplus S(M, A^1_0) \otimes \operatorname{Hom}(N, B^1_p).$$

Let us analyse the case $p = 0$. Under the identification given by Proposition 3.4, the map

$$\iota'_0 : \mathrm{rad}(M, A^1_0) \otimes \mathrm{rad}(N, B^1_0) \to \mathrm{rad}(M \otimes N, A^1_0 \otimes B^1_0)$$

is the identity if $M \ncong A^1_0$ and $N \ncong B^1_0$, and the inclusion otherwise. If $M \ncong A^1_0$ and $N \cong B^1_0$, then we are in the same situation as in the previous case, and

$$\operatorname{coker} \iota'_0 = \mathrm{Hom}(M, A^1_0) \otimes S(B^1_0)$$

and similarly for the symmetric case. If both $M \cong A^1_0$ and $N \cong B^1_0$, then we claim that

$$\operatorname{coker} \iota'_0 = \mathrm{rad}(A^1_0) \otimes S(B^1_0) \oplus S(A^1_0) \otimes \mathrm{rad}(B^1_0).$$

Indeed (for simplicity, write $E = A^1_0$ and $F = B^1_0$), in the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{rad}(E) \otimes \mathrm{rad}(F) \arrow[r, "{\left[\begin{smallmatrix} \alpha \\ -\alpha \end{smallmatrix}\right]}"] \arrow[d, equal] & \begin{smallmatrix} \mathrm{rad}(E)\otimes\mathrm{rad}(F) \\ \oplus \\ \mathrm{rad}(E)\otimes\mathrm{rad}(F) \end{smallmatrix} \arrow[r, "{\left[\begin{smallmatrix} \alpha & \alpha \end{smallmatrix}\right]}"] \arrow[d, "{\left[\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\right]}"] & \mathrm{rad}(E) \otimes \mathrm{rad}(F) \arrow[r] \arrow[d, "\iota'_0"] & 0 \\
0 \arrow[r] & \mathrm{rad}(E) \otimes \mathrm{rad}(F) \arrow[r, "{\left[\begin{smallmatrix} \alpha \\ -\alpha \end{smallmatrix}\right]}"] \arrow[d] & \begin{smallmatrix} \mathrm{rad}(E)\otimes\mathrm{End}(F) \\ \oplus \\ \mathrm{End}(E)\otimes\mathrm{rad}(F) \end{smallmatrix} \arrow[r, "{\left[\begin{smallmatrix} \alpha & \alpha \end{smallmatrix}\right]}"] \arrow[d] & \mathrm{rad}(E \otimes F) \arrow[r] \arrow[d] & 0 \\
 & 0 & \begin{smallmatrix} \mathrm{rad}(E)\otimes S(F) \\ \oplus \\ S(E)\otimes\mathrm{rad}(F) \end{smallmatrix} \arrow[d] & \operatorname{coker} \iota'_0 \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

the first row is exact, as well as all the columns (α denotes the canonical map $f \otimes g \mapsto f \otimes g$). The second row is exact by Lemma 3.6. Hence we get an isomorphism

$$\mathrm{rad}(E) \otimes S(F) \oplus S(E) \otimes \mathrm{rad}(F) \cong \operatorname{coker} \iota'_0$$

by the 3×3 lemma. We have shown that

$$\operatorname{coker} \iota'_p = F_M(A^1_p) \otimes S(N, B^1_0) \oplus S(M, A^1_0) \otimes F_N(B^1_p)$$

for every value of $p = 0, \dots, m+n$.

It remains to show that the differentials $\operatorname{coker} \iota'_{p+1} \to \operatorname{coker} \iota'_p$ are diagonal, to conclude that the direct-sum decomposition of the objects is actually a direct-sum decomposition into the two complexes appearing in equation (1). The only degree where this poses problems is $p = 0$ in the case $M \cong E = A^1_0,\ N \cong F = B^1_0$. For this, consider the following diagram:

$$\begin{tikzcd}
\begin{smallmatrix}\mathrm{Hom}(E,A_1^1)\otimes\mathrm{rad}(F)\\ \bigoplus \\ \mathrm{rad}(E)\otimes\mathrm{Hom}(F,B_1^1)\end{smallmatrix} \arrow[r, "\beta"] \arrow[d, "\iota_1'"'] & \mathrm{rad}(E)\otimes\mathrm{rad}(F) \arrow[d, "\iota_0'"] \\
\begin{smallmatrix}\mathrm{Hom}(E,A_1^1)\otimes\mathrm{End}(F)\\ \bigoplus \\ \mathrm{End}(E)\otimes\mathrm{Hom}(F,B_1^1)\end{smallmatrix} \arrow[r, "\beta"] \arrow[d] & \mathrm{rad}(E\otimes F) \arrow[d] \\
\begin{smallmatrix}\mathrm{Hom}(E,A_1^1)\otimes S(F)\\ \bigoplus \\ S(E)\otimes\mathrm{Hom}(F,B_1^1)\end{smallmatrix} \arrow[r] & \operatorname{coker}\iota_0'
\end{tikzcd}$$

where the horizontal maps are induced by

$$\beta = \left[(d_1^A\circ -)\otimes\mathrm{id},\quad \mathrm{id}\otimes(d_1^B\circ -)\right],$$

which is the last map appearing in the sequence $F_E(A_\bullet^1)\otimes^T F_F(B_\bullet^1)$. The map β factors as

$$\beta = \begin{bmatrix}\alpha & \alpha\end{bmatrix}\begin{bmatrix}(d_1^A\circ -)\otimes\mathrm{id} & 0\\ 0 & \mathrm{id}\otimes(d_1^B\circ -)\end{bmatrix}$$

hence the diagram above can be completed to a diagram

$$\begin{tikzcd}
\begin{smallmatrix}\mathrm{Hom}(E,A_1^1)\otimes\mathrm{rad}(F)\\ \bigoplus \\ \mathrm{rad}(E)\otimes\mathrm{Hom}(F,B_1^1)\end{smallmatrix} \arrow[r] \arrow[d, "\iota_1'"'] & \begin{smallmatrix}\mathrm{rad}(E)\otimes\mathrm{rad}(F)\\ \bigoplus \\ \mathrm{rad}(E)\otimes\mathrm{rad}(F)\end{smallmatrix} \arrow[r, "{[\alpha\ \alpha]}"] \arrow[d] & \mathrm{rad}(E)\otimes\mathrm{rad}(F) \arrow[d, "\iota_0'"] \\
\begin{smallmatrix}\mathrm{Hom}(E,A_1^1)\otimes\mathrm{End}(F)\\ \bigoplus \\ \mathrm{End}(E)\otimes\mathrm{Hom}(F,B_1^1)\end{smallmatrix} \arrow[r] \arrow[d] & \begin{smallmatrix}\mathrm{rad}(E)\otimes\mathrm{End}(F)\\ \bigoplus \\ \mathrm{End}(E)\otimes\mathrm{rad}(F)\end{smallmatrix} \arrow[r, "{[\alpha\ \alpha]}"] \arrow[d] & \mathrm{rad}(E\otimes F) \arrow[d] \\
\begin{smallmatrix}\mathrm{Hom}(E,A_1^1)\otimes S(F)\\ \bigoplus \\ S(E)\otimes\mathrm{Hom}(F,B_1^1)\end{smallmatrix} \arrow[r] & \begin{smallmatrix}\mathrm{rad}(E)\otimes S(F)\\ \bigoplus \\ S(E)\otimes\mathrm{rad}(F)\end{smallmatrix} \arrow[r, "\cong"] & \operatorname{coker}\iota_0'
\end{tikzcd},$$

where the horizontal maps on the left-hand side are diagonal. Hence the induced map

$$\begin{smallmatrix}\mathrm{Hom}(E,A_1^1)\otimes S(F)\\ \bigoplus \\ S(E)\otimes\mathrm{Hom}(F,B_1^1)\end{smallmatrix} \longrightarrow \operatorname{coker}\iota_0'$$

factors through the diagonal map

$$\begin{bmatrix}(d_1^A\circ -)\otimes\mathrm{id}_{S(F)} & 0\\ 0 & \mathrm{id}_{S(E)}\otimes(d_1^B\circ -)\end{bmatrix} \quad : \quad \begin{smallmatrix}\mathrm{Hom}(E,A_1^1)\otimes S(F)\\ \bigoplus \\ S(E)\otimes\mathrm{Hom}(F,B_1^1)\end{smallmatrix} \longrightarrow \begin{smallmatrix}\mathrm{rad}(E)\otimes S(F)\\ \bigoplus \\ S(E)\otimes\mathrm{rad}(F)\end{smallmatrix}$$

and we are done. $\square$

4. Examples

As an example, consider the quiver

$$1 \longrightarrow 2 \longrightarrow 3 \longleftarrow 4 \longleftarrow 5$$

and the corresponding path algebra $A = kQ$. Thus A is 3-homogeneous, (1-) representation finite (see [2,6]). We want to consider the algebra $B = A \otimes A$, which is then 3-homogeneous, 2-representation finite. There are 15 nonisomorphic indecomposables in $\operatorname{mod} A$, which have the following dimension vectors:

$$\begin{array}{lll} P_1 : (11100) & M_1 : (01111) & I_1 : (10000) \\ P_2 : (01100) & M_2 : (01000) & I_2 : (11000) \\ P_3 : (00100) & M_3 : (01110) & I_3 : (11111) \\ P_4 : (00110) & M_4 : (00010) & I_4 : (00011) \\ P_5 : (00111) & M_5 : (11110) & I_5 : (00001). \end{array}$$

The Auslander–Reiten quiver of A is the following:

$$\begin{tikzcd}
 & & P_1 \arrow[dr] \arrow[rr, dotted, no head] & & M_4 \arrow[dr] \arrow[rr, dotted, no head] & & I_5 \\
 & P_2 \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & M_5 \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & I_4 \arrow[ur] & \\
P_3 \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & M_3 \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & I_3 \arrow[ur] \arrow[dr] & & \\
 & P_4 \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & M_1 \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & I_2 \arrow[dr] & \\
 & & P_5 \arrow[ur] \arrow[rr, dotted, no head] & & M_2 \arrow[ur] \arrow[rr, dotted, no head] & & I_1
\end{tikzcd}$$

where the dotted lines represent τ_1^-.

Inside $\operatorname{mod} B$ we have the 2-cluster tilting subcategory $\mathcal{C} = \operatorname{add} M$, where

$$M = \bigoplus_{1 \le i,j \le 5} P_i \otimes P_j \oplus \bigoplus_{1 \le i,j \le 5} M_i \otimes M_j \oplus \bigoplus_{1 \le i,j \le 5} I_i \otimes I_j.$$

Let us consider for instance the (1-)almost split sequences

$$C_\bullet = \; 0 \longrightarrow P_2 \xrightarrow{\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]} P_1 \oplus M_3 \xrightarrow{\left[\begin{smallmatrix} c & d \end{smallmatrix}\right]} M_5 \longrightarrow 0$$

and

$$D_\bullet = \; 0 \longrightarrow P_5 \xrightarrow{e} M_1 \xrightarrow{f} M_2 \longrightarrow 0$$

in $\operatorname{mod} A$. Notice that both these sequences start in slice 0. The sequence $C_\bullet$ is isomorphic to the cone of

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P_2 \arrow[r, "-a"] \arrow[d, "-b"'] & P_1 \arrow[r] \arrow[d, "-c"'] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & M_3 \arrow[r, "d"] & M_5 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

and $D_\bullet$ is isomorphic to the cone of

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P_5 \arrow[r] \arrow[d, "-e"'] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & M_1 \arrow[r, "f"] & M_2 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

where these diagrams should be seen as morphisms φ, ψ of chain complexes. Then we can construct the morphism $\varphi \otimes^T \psi$:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P_2 \otimes P_5 \arrow[r, "-a\otimes 1"] \arrow[d, "b\otimes e"'] & P_1 \otimes P_5 \arrow[r] \arrow[d, "\begin{bmatrix} 0 \\ c\otimes e \end{bmatrix}"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & M_3 \otimes M_1 \arrow[r, "\begin{bmatrix} -1\otimes f \\ d\otimes 1 \end{bmatrix}"] & \begin{matrix} M_3\otimes M_2 \\ \oplus \\ M_5\otimes M_1 \end{matrix} \arrow[r, "\begin{bmatrix} d\otimes 1 & 1\otimes f \end{bmatrix}"] & M_5 \otimes M_2 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

The cone $E_\bullet = \mathrm{Cone}(\varphi \otimes^T \psi)$ is then the sequence

$$0 \longrightarrow P_2 \otimes P_5 \xrightarrow{\begin{bmatrix} a\otimes 1 \\ -b\otimes e \end{bmatrix}} \begin{matrix} P_1\otimes P_5 \\ \oplus \\ M_3\otimes M_1 \end{matrix} \xrightarrow{\begin{bmatrix} 0 & -1\otimes f \\ -c\otimes e & d\otimes 1 \end{bmatrix}} \begin{matrix} M_3\otimes M_2 \\ \oplus \\ M_5\otimes M_1 \end{matrix} \xrightarrow{\begin{bmatrix} d\otimes 1 & 1\otimes f \end{bmatrix}} M_5 \otimes M_2 \longrightarrow 0$$

which is 2-almost split in $\mathcal{C}$ by Theorem 1.1.

Now we can go further, and consider the algebra $B \otimes A$, which is then 3-homogeneous, 3-representation finite. Let us write for simplicity $P_{abc} = P_a \otimes P_b \otimes P_c$ and $M_{abc} = M_a \otimes M_b \otimes M_c$. We look at the 3-almost split sequence starting in P_{254}, which is obtained from $E_\bullet$ together with the sequence

$$0 \longrightarrow P_4 \longrightarrow P_5 \oplus M_3 \longrightarrow M_1 \longrightarrow 0$$

in mod A. By applying the formula we get the sequence

$$\begin{tikzcd}
 & & P_{155} \arrow[dr] & & \\
 & P_{154} \arrow[ur] \arrow[dr] & \oplus & M_{511} \arrow[ddr] & \\
 & \oplus & M_{513} \arrow[ur] \arrow[dr] & \oplus & \\
P_{254} \arrow[uur] \arrow[r] \arrow[ddr] & P_{255} \arrow[uuur] \arrow[dr] & \oplus & M_{523} \arrow[r] & M_{521} \\
 & \oplus & M_{311} \arrow[uuur] \arrow[dr] & \oplus & \\
 & M_{313} \arrow[uuur] \arrow[ur] \arrow[dr] & \oplus & M_{321} \arrow[uur] & \\
 & & M_{323} \arrow[uuur] \arrow[ur] & &
\end{tikzcd}$$

where each arrow is the natural morphism up to sign.

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Paper II

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Tensor products of n-complete algebras

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ABSTRACT

If A and B are n- and m-representation finite k-algebras, then their tensor product $\Lambda = A \otimes_k B$ is not in general $(n+m)$-representation finite. However, we prove that if A and B are acyclic and satisfy the weaker assumption of n- and m-completeness, then Λ is $(n+m)$-complete. This mirrors the fact that taking higher Auslander algebra does not preserve d-representation finiteness in general, but it does preserve d-completeness. As a corollary, we get the necessary condition for Λ to be $(n+m)$-representation finite which was found by Herschend and Iyama by using a certain twisted fractionally Calabi–Yau property.

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1. Introduction

Higher Auslander–Reiten theory was developed in a series of papers [10], [9], [11] as a tool to study module categories of finite-dimensional algebras. The idea is to replace all the homological notions in classical Auslander–Reiten theory with higher-dimensional analogs. Some early results can be found in [8], [5]. This approach has been fruitful in the context of noncommutative algebraic geometry, see for instance [1], [7], [6]. Higher Auslander–Reiten theory is also deeply tied with d-homological algebra ([3], [13], [15]). A presentation of the theory from this point of view can be found in [14].

In this setting, d-representation finite algebras were introduced in [12] as a generalisation of hereditary representation finite algebras. They are algebras of global dimension at most d that have a d-cluster tilting module M. The category $\operatorname{add} M$ has nice homological properties and behaves in many ways like the module category of a hereditary representation finite algebra. While classification of d-representation finite algebras seems far from being achieved, it makes sense to look for examples, and to try to understand how d-representation finiteness behaves with respect to reasonable operations. Notice that in this setting we have more freedom than in the hereditary case, since we are allowed to increase the global dimension and still fall within the scope of the theory.

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For instance, in [12] Iyama investigates whether the endomorphism algebra of the d-cluster tilting module (called the higher Auslander algebra) is $(d+1)$-representation finite. This turns out to be false in general, but a necessary and sufficient condition is given: the only case where it is true is within the tower of iterated higher Auslander algebras of the upper triangular matrix algebra, so this construction gives only a specific family of examples. On the other hand, in the same paper the weaker notion of d-complete algebra is introduced and studied. A d-complete algebra is an algebra of global dimension at most d that has a module which is d-cluster tilting in a suitable exact subcategory of the module category. It turns out that this weaker notion is preserved under taking higher Auslander algebras, thereby producing many examples of d-complete algebras for any d.

Another operation one might investigate is that of taking tensor products over the base field k. Indeed, if k is perfect then $\operatorname{gl.dim} A \otimes_k B = \operatorname{gl.dim} A + \operatorname{gl.dim} B$, so it makes sense to ask whether the tensor product of an n- and an m-representation finite algebras is $(n+m)$-representation finite. This is false in general, and in [4] Herschend and Iyama give a necessary and sufficient condition (l-homogeneity) for it to be true.

In this paper we prove that the same weaker notion of d-completeness which is used in [12] is preserved under tensor products, under the assumption of acyclicity. Namely, if A is n-complete and acyclic and B is m-complete and acyclic, then $A \otimes_k B$ is $(n+m)$-complete and acyclic. If we assume that A and B are l-homogeneous, we recover the result by Herschend and Iyama. This gives a new way of producing d-complete algebras for any d.

The proof we give is structured as follows. We prove that over the tensor product there are $(n+m)$-almost split sequences (using the same construction as in [16]), and moreover that injective modules have source sequences. Then we use these sequences, combined with the assumption of acyclicity, to prove that the module T in the definition of $(n+m)$-completeness is tilting. By [12, Theorem 2.2(b)], the existence of the above sequences in $T^{\perp}$ is equivalent to M being $(n+m)$-cluster tilting in $T^{\perp}$, which is the key point of $(n+m)$-completeness.

In Section 2 we lay down notation, conventions, and preliminary definitions. Section 3 contains the statement of our main result. Section 4 contains the results about d-almost split sequences and tensor products which we want to use. Section 5 is dedicated to proving the main theorem, which amounts to checking that the tensor product satisfies the defining properties of $(n+m)$-complete algebras. In Section 6 we present some examples.

2. Notation and conventions

Throughout this paper, k denotes a fixed perfect field. All algebras are associative, unital, and finite dimensional over k. For an algebra Λ, $\operatorname{mod}\Lambda$ (respectively $\Lambda\operatorname{mod}$) denotes the category of finitely generated right (left) Λ-modules. We denote by D the duality $D = \operatorname{Hom}_k(-,k)$ between $\operatorname{mod}\Lambda$ and $\Lambda\operatorname{mod}$ (in both directions). Subcategories are always assumed to be full and closed under isomorphisms, finite direct sums and summands. For $M \in \operatorname{mod}\Lambda$, we denote by $\operatorname{add} M$ the subcategory of $\operatorname{mod}\Lambda$ whose objects are all modules isomorphic to finite direct sums of summands of M. We write $\operatorname{rad}_\Lambda(-,-)$ for the subfunctor of $\operatorname{Hom}_\Lambda(-,-)$ defined by

$$\operatorname{rad}_\Lambda(X,Y) = \{f \in \operatorname{Hom}_\Lambda(X,Y) \mid \operatorname{id}_X - g \circ f \text{ is invertible } \forall g \in \operatorname{Hom}_\Lambda(Y,X)\}.$$

Moreover, for $X, Y \in \operatorname{mod}\Lambda$, we write $\operatorname{top}_\Lambda(X,Y) = \operatorname{Hom}_\Lambda(X,Y)/\operatorname{rad}_\Lambda(X,Y)$. We often write Hom instead of $\operatorname{Hom}_\Lambda$ and similarly for rad and top when the context allows it. We denote by $\mathcal{D}^b(\Lambda)$ the bounded derived category of $\operatorname{mod}\Lambda$. For a subcategory $\mathcal{C}$ of $\mathcal{D}^b(\Lambda)$, we denote by $\operatorname{thick}\mathcal{C}$ the smallest triangulated subcategory of $\mathcal{D}^b(\Lambda)$ containing $\mathcal{C}$. If $\mathcal{C} = \operatorname{add} M$ for some $M \in \operatorname{mod}\Lambda \subseteq \mathcal{D}^b(\Lambda)$, we write $\operatorname{thick} M = \operatorname{thick}(\operatorname{add} M)$. All tensor products are over k, even when the specification is omitted to simplify the notation.

Throughout this section, let $\operatorname{gl.dim}\Lambda \leq d$. Then we can define the *higher Auslander–Reiten translations* by

$$\tau_d = D\operatorname{Ext}^d_\Lambda(-,\Lambda) : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda$$
$$\tau_d^- = \operatorname{Ext}^d_{\Lambda^{op}}(D-,\Lambda) : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda.$$

We are interested in categories associated to tilting modules.

Definition 2.1. A Λ-module T is *tilting* if the following conditions are satisfied:

(1) $\operatorname{Ext}^i(T,T) = 0$ for all $i \neq 0$,
(2) there is an exact sequence $0 \to \Lambda \to T_0 \to \cdots \to T_m \to 0$ for some m with $T_i \in \operatorname{add} T$ for all i.

The second condition in the definition can be replaced by

$$\operatorname{thick} T = \mathcal{D}^b(\Lambda).$$

For a tilting module T, we have an exact subcategory of $\operatorname{mod}\Lambda$

$$T^\perp = \left\{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i(T,X) = 0 \text{ for every } i \neq 0\right\}$$

We are interested in d-cluster tilting subcategories of $T^\perp$.

Definition 2.2. Let T be a tilting module. A subcategory $\mathcal{C}$ of $T^\perp$ is called *d-cluster tilting* if

$$\mathcal{C} = \left\{X \in T^\perp \mid \operatorname{Ext}^i(\mathcal{C},X) = 0 \text{ for every } 0 < i < d\right\} =$$
$$= \left\{X \in T^\perp \mid \operatorname{Ext}^i(X,\mathcal{C}) = 0 \text{ for every } 0 < i < d\right\}.$$

We follow [12, Definition 1.11] and define the following subcategories of $\operatorname{mod}\Lambda$:

(1) $\mathcal{M} = \mathcal{M}(\Lambda) = \operatorname{add}\left\{\tau_d^i D\Lambda \mid i \geq 0\right\}$,
(2) $\mathcal{P} = \{X \in \mathcal{M} \mid \tau_d X = 0\}$,
(3) $\mathcal{M}_P = \{X \in \mathcal{M} \mid X \text{ has no nonzero summands in } \mathcal{P}\}$,
(4) $\mathcal{M}_I = \{X \in \mathcal{M} \mid X \text{ has no nonzero summands in } \operatorname{add} D\Lambda\}$.

Let T_Λ be a basic module such that $\operatorname{add} T_\Lambda = \mathcal{P}$.

Definition 2.3. An algebra Λ is *d-complete* if the following conditions hold:

(A_d) T_Λ is a tilting module.
(B_d) $\mathcal{M}$ is a d-cluster tilting subcategory of $T_\Lambda^\perp$.
(C_d) $\operatorname{Ext}^i(\mathcal{M}_P,\Lambda) = 0$ for every $0 < i < d$.

Note that condition (A_d) implies that $\tau_d^l = 0$ for large l ([12, Proposition 1.12(d) and 1.3(c)]). Note moreover that if Λ is d-complete then since $\operatorname{gl.dim}\Lambda \leq d$ it follows that $\operatorname{gl.dim}\Lambda \in \{0,d\}$. This is a generalisation of the notion of d-representation finiteness which we use in [16]. Without loss of generality, from now on we assume that Λ is basic. We write T for T_Λ when the context allows it. Then [12, Proposition 1.13] says that "d-representation finite" is the same as "d-complete with $T = \Lambda$".

If Λ is d-complete, then for every indecomposable injective I_i there is a unique $l_i \in \mathbb{N}$ such that $\tau_d^{l_i-1} I_i \in \mathcal{P}$, and

$$T_\Lambda = \bigoplus_i \tau_d^{l_i-1} I_i.$$

Definition 2.4 *([4]).* Let Λ be a k-algebra of global dimension d such that $\tau_d^l = 0$ for l sufficiently large. We say that Λ is *l-homogeneous* if $\tau_d^{l-1} D\Lambda = T_\Lambda$.

If Λ is d-complete, this means that $l_i = l$ for every i.

Our main result is proved only for acyclic algebras, let us define what we mean by that. Let $M \in \operatorname{mod}\Lambda$, and let $\mathcal{C} = \operatorname{add} M$. We want to define a preorder on the indecomposable objects $\operatorname{ind}\mathcal{C}$ of $\mathcal{C}$. For $X, Y \in \operatorname{ind}\mathcal{C}$, we say $X < Y$ if there is a sequence $(X = X_0, X_1, \dots, X_{m+1} = Y)$ for some $m \geq 0$, such that $X_i \in \operatorname{ind}\mathcal{C}$ and $\operatorname{rad}_\Lambda(X_i, X_{i+1}) \neq 0$ for all i. This defines a transitive relation $<$ on $\operatorname{ind}\mathcal{C}$. Notice that we can replace $\operatorname{rad}_\Lambda(X_i, X_{i+1}) \neq 0$ with $\operatorname{rad}_\mathcal{C}(X_i, X_{i+1}) \neq 0$ above.

Definition 2.5. The category $\mathcal{C}$ is *directed* if $<$ is antisymmetric, that is if no indecomposable module $X \in \mathcal{C}$ satisfies $X < X$. If $\mathcal{C} = \operatorname{add} M$, we say that M is directed. We call the algebra Λ *acyclic* if Λ_Λ is directed.

3. Main result

We now consider the case where A is n-complete, B is m-complete, and $\Lambda = A \otimes_k B$. Since k is perfect, we have that $\operatorname{gl.dim}\Lambda = \operatorname{gl.dim} A + \operatorname{gl.dim} B$. Moreover, by the Künneth formula we have $\tau_{n+m} X \otimes Y = \tau_n X \otimes \tau_m Y$. Since indecomposable injective Λ-modules are of the form $X \otimes Y$, it follows that all indecomposable modules in $\mathcal{M}$ are of this form. Our main result is the following:

Theorem 3.1. *Let A, B be n- respectively m-complete acyclic k-algebras, with k perfect. Then $A \otimes_k B$ is $(n+m)$-complete and acyclic.*

Note that as far as the author is aware, there are no known examples of d-complete algebras which are not acyclic (this is Question 5.9 in [7]).

This result can be applied inductively to construct d-complete algebras starting for example from hereditary representation finite algebras and taking tensor products. In this sense, it is similar in spirit to [12, Theorem 1.14 and Corollary 1.16], where Iyama constructs towers of d-complete algebras (with increasing d) by taking iterated higher Auslander algebras. The algebra $A \otimes B$ is almost never $(n+m)$-representation finite by the characterisation given by Herschend and Iyama in [4]. Our result specialises to their characterisation in the acyclic case:

Corollary 3.2. *Let A, B be n- respectively m-representation finite acyclic k-algebras, with k perfect. Then the following are equivalent:*

(1) *$A \otimes_k B$ is $(n+m)$-representation finite.*
(2) *$\exists l \in \mathbb{N}$ such that A and B are l-homogeneous.*

Moreover, in this case $A \otimes_k B$ is also l-homogeneous.

It should be noted that there is a choice involved in the definition we gave of d-completeness, namely that we take $\mathcal{M}$ to be the τ_d-completion of $\operatorname{add} D\Lambda$. We might as well take $\mathcal{M}$ to be the τ_d^--completion of $\operatorname{add}\Lambda$, and call Λ d-cocomplete if it satisfies the dual conditions to $(A_d), (B_d), (C_d)$. Then Λ is d-complete if and only if Λ^{op} is d-cocomplete. Notice that d-representation finite is the same as d-complete and d-cocomplete with the same $\mathcal{M}$. However, if A and B are n- and m-representation finite, then $A \otimes B$ is $(n+m)$-complete and cocomplete, but in general not with the same $\mathcal{M}$.

4. Preparation

4.1. d-complete algebras

Following [12], we make some observations about d-complete algebras in general. Fix a finite-dimensional algebra Λ.

Lemma 4.1. *If* $\operatorname{gl.dim} \Lambda \leq d$*, the following are equivalent:*

(1) $\operatorname{Ext}^i(\mathcal{M}_P, \Lambda) = 0$ *for* $0 < i < d$
(2) $\operatorname{Ext}^i(\mathcal{M}_P, \Lambda) = 0$ *for* $0 \leq i < d$.

Proof. The only direction to prove follows from [12, Lemma 2.3(b)]. □

Proposition 4.2. *If* Λ *is* d*-complete, then*

$$\operatorname{Hom}(\tau_d^i D\Lambda, \tau_d^j D\Lambda) = 0$$

if $i < j$.

Proof. This follows from [12, Lemma 2.4(e)]. □

We can define *slices* $\mathcal{S}(i)$ on $\mathcal{M}$ by saying that $\mathcal{S}(i) = \operatorname{add} \tau_d^i D\Lambda$. Thus

$$\mathcal{M} = \bigvee_{i \geq 0} \mathcal{S}(i)$$

(meaning that every object $X \in \mathcal{M}$ can be written uniquely as $X = \bigoplus_{i \geq 0} X_i$ with $X_i \in \mathcal{S}(i)$) and moreover $\operatorname{Hom}(\mathcal{S}(i), \mathcal{S}(j)) = 0$ if $i < j$.

Lemma 4.3. *If* Λ *is* d*-complete then* $\tau_d^{\pm}$ *induce quasi-inverse equivalences* $\mathcal{M}_P \leftrightarrow \mathcal{M}_I$.

Proof. This is [12, Lemma 2.4(b)]. □

4.2. d-almost split sequences

In the spirit of generalising Auslander–Reiten theory, it is natural to define the higher analog of almost split sequences as follows.

Definition 4.1 *(Iyama).* A complex with objects in a subcategory $\mathcal{C}$ of $\operatorname{mod} \Lambda$

$$C_d \xrightarrow{f_d} C_{d-1} \xrightarrow{f_{d-1}} C_{d-2} \xrightarrow{f_{d-2}} \cdots$$

is a *source sequence (in* $\mathcal{C}$*) of* C_d if the following conditions hold:

(1) $f_i \in \mathrm{rad}(C_i, C_{i-1})$ for all i,
(2) the sequence of functors

$$\cdots \xrightarrow{-\circ f_{d-2}} \mathrm{Hom}(C_{d-2}, -) \xrightarrow{-\circ f_{d-1}} \mathrm{Hom}(C_{d-1}, -) \xrightarrow{-\circ f_d} \mathrm{rad}(C_d, -) \longrightarrow 0$$

is exact on $\mathcal{C}$.

Dually we can define *sink sequences*. An exact sequence

$$0 \longrightarrow C_{d+1} \longrightarrow C_{d-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$$

is an *d-almost split sequence* if it is a source sequence of C_{d+1} and a sink sequence of C_0. We say that such d-almost split sequence starts in C_{d+1} and ends in C_0.

Definition 4.2. We say that $\mathcal{M} = \mathcal{M}(\Lambda)$ *has d-almost split sequences* if for every indecomposable $X \in \mathcal{M}_I$ (respectively $Y \in \mathcal{M}_P$) there is an d-almost split sequence in $\mathcal{C}$

$$0 \to X \to C_d \to \cdots \to C_1 \to Y \to 0.$$

In this case we must have $X \cong \tau_d Y, Y \cong \tau_d^- X$. This holds for d-complete algebras ([12, Theorem 2.2(a)(i)]):

Theorem 4.4. *If Λ is d-complete, then $\mathcal{M}$ has d-almost split sequences.*

To apply the methods introduced in [16], we need to rephrase Definition 4.1 as follows: for any indecomposable $X \in \mathcal{C}$ we can define a functor F_X on complexes of radical maps by mapping

$$C_\bullet = \ \cdots \xrightarrow{f_{i+1}} C_i \xrightarrow{f_i} \cdots \xrightarrow{f_1} C_0 \xrightarrow{f_0} \cdots$$

to

$$F_X(C_\bullet) = \ \cdots \xrightarrow{f_{i+1}\circ -} \mathrm{Hom}(X, C_i) \xrightarrow{f_i \circ -} \cdots \xrightarrow{f_1 \circ -} \mathrm{rad}(X, C_0) \xrightarrow{f_0 \circ -} \cdots$$

(that is, F_X is the subfunctor of $\mathrm{Hom}(X, -)$ given by replacing $\mathrm{Hom}(X, C_0)$ with $\mathrm{rad}(X, C_0)$). Similarly, we can define a subfunctor G_X of the contravariant functor $\mathrm{Hom}(-, X)$ by mapping $C_\bullet$ to

$$G_X(C_\bullet) = \ \cdots \xrightarrow{-\circ f_0} \mathrm{Hom}(C_0, X) \xrightarrow{-\circ f_1} \cdots \xrightarrow{-\circ f_{d+1}} \mathrm{rad}(C_{d+1}, X) \xrightarrow{-\circ f_{d+2}} \cdots$$

Lemma 4.5. *Let $C_\bullet$ be a complex in $\mathcal{C}$. Then*

(1) *If $C_i = 0$ for all $i > d+1$, then $C_\bullet$ is a sink sequence if and only if $F_X(C_\bullet)$ is exact for every $X \in \mathcal{C}$.*
(2) *If $C_i = 0$ for all $i < 0$, then $C_\bullet$ is a source sequence if and only if $G_X(C_\bullet)$ is exact for every $X \in \mathcal{C}$.*
(3) *If $C_i = 0$ for all $i > d+1$ and $i < 0$, then $C_\bullet$ is d-almost split if and only if $F_X(C_\bullet)$ and $G_X(C_\bullet)$ are exact for every $X \in \mathcal{C}$.*

Proof. Direct check using the definitions. □

By additivity, in the above Lemma we can replace "every $X \in \mathcal{C}$" by "every indecomposable $X \in \mathcal{C}$".

Notice that since d-almost split sequences come from minimal projective resolutions of a functor $\operatorname{rad}(C_0, -)$, they are uniquely determined by C_0 up to isomorphism of complexes. Moreover, we have

Lemma 4.6. *Any map $f_0 : C_0 \to D_0$ between indecomposables in $\mathcal{M}_P$ induces a map of complexes $f_\bullet : C_\bullet \to D_\bullet$, where*

$$C_\bullet = \qquad 0 \longrightarrow C_{d+1} \xrightarrow{g_{d+1}} \cdots \xrightarrow{g_1} C_0 \longrightarrow 0,$$

$$D_\bullet = \qquad 0 \longrightarrow D_{d+1} \xrightarrow{h_{d+1}} \cdots \xrightarrow{h_1} D_0 \longrightarrow 0$$

are the d-almost split sequences ending in C_0 and D_0 respectively, if these exist.

Proof. The map $f_0 g_1 : C_1 \to D_0$ is a radical morphism, and since

$$\operatorname{Hom}(C_1, D_1) \xrightarrow{h_1 \circ -} \operatorname{rad}(C_1, D_0)$$

is surjective, there is a map $f_1 : C_1 \to D_1$ such that $h_1 f_1 = f_0 g_1$. Now assume we have constructed maps $f_j : C_j \to D_j$ that make all diagrams commute, for all $0 \leq j < i$ for some $i \geq 2$. We have that

$$\operatorname{Hom}(C_i, D_i) \xrightarrow{h_i \circ -} \operatorname{Hom}(C_i, D_{i-1}) \xrightarrow{h_{i-1} \circ -} \operatorname{Hom}(C_i, D_{i-2})$$

is exact in the middle term by assumption. Since $h_{i-1} f_{i-1} g_i = f_{i-2} g_{i-1} g_i = 0$, we have that $f_{i-1} g_i \in \ker(h_{i-1} \circ -) = \operatorname{im}(h_i \circ -)$, that is there is a map $f_i : C_i \to D_i$ such that $f_{i-1} g_i = h_i f_i$. The f_i's we have defined recursively give by construction a map of complexes $f_\bullet : C_\bullet \to D_\bullet$. $\square$

The following is a result which appeared in [16] in the setting of d-representation finite algebras, and which can be reformulated in the setting of d-complete algebras.

Theorem 4.7. *Let Λ be d-complete. Let $X \in \mathcal{S}(i)$ with $i > 0$. Then the d-almost split sequence starting in X is isomorphic as a complex to* $\operatorname{Cone} \varphi$*, where $\varphi : E_\bullet \to F_\bullet$ is a map of complexes, such that:*

(1) *All the maps appearing in $E_\bullet$, $F_\bullet$, and the components of φ are radical,*
(2) *$E_j \in \mathcal{S}(i)$ and $F_j \in \mathcal{S}(i-1)$ for every j.*

Proof. This is shown exactly as in [16, Theorem 2.3]. Namely, one decomposes the modules M_j appearing in the d-almost split sequence starting in X as $M_j = \bigoplus_{i \geq 0} M_{ij}$ with $M_{ij} \in \mathcal{S}(i)$. One checks using Proposition 4.2 that in order for the sequence to be d-almost split, all the M_j must be in $\operatorname{add}\left(\tau_d^i D\Lambda \oplus \tau_d^{i-1} D\Lambda\right)$ for some i. Now let $E_j = M_{i,j+1}$ and $F_j = M_{i-1,j}$. Using that $\operatorname{Hom}(\tau_d^{i-1} D\Lambda, \tau_d^i D\Lambda) = 0$ one can choose suitable differentials for $E_\bullet$ and $F_\bullet$ and a morphism $\varphi_\bullet : E_\bullet \to F_\bullet$ such that $\operatorname{Cone} \varphi$ is the desired sequence. $\square$

We will need a technical lemma:

Lemma 4.8. *Let*

$$0 \longrightarrow C_{d+1} \xrightarrow{f_{d+1}} C_d \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{f_1} C_0 \longrightarrow 0$$

be a d-almost split sequence. Then for any choice of decomposition of the modules C_i into indecomposables, the corresponding matrices of the maps f_i have no zero column and no zero row.

Proof. We argue by contradiction. Assume f_i has a zero column for $i > 1$. Then there is a complex

$$C_{i+1} \xrightarrow{\begin{bmatrix} f_{i+1}^1 \\ f_{i+1}^2 \end{bmatrix}} C_i^1 \oplus C_i^2 \xrightarrow{\begin{bmatrix} f_i^1 & 0 \end{bmatrix}} C_{i-1}$$

such that

$$\operatorname{Hom}(C_i^2, C_{i+1}) \xrightarrow{\begin{bmatrix} f_{i+1}^1 \circ - \\ f_{i+1}^2 \circ - \end{bmatrix}} \begin{matrix} \operatorname{Hom}(C_i^2, C_i^1) \\ \oplus \\ \operatorname{Hom}(C_i^2, C_i^2) \end{matrix} \xrightarrow{\begin{bmatrix} f_i^1 \circ - & 0 \end{bmatrix}} \operatorname{Hom}(C_i^2, C_{i-1})$$

is exact in the middle, which implies that $f_{i+1}^2 \circ -$ is surjective on $\operatorname{Hom}(C_i^2, C_i^2)$, and so there is $h \in \operatorname{Hom}(C_i^2, C_{i+1})$ such that $f_{i+1}^2 \circ h = \mathrm{id}_{C_i^2}$. Since $f_{i+1}^2 \in \operatorname{rad}(C_{i+1}, C_i)$, it follows that $C_i^2 = 0$ and we are done. For proving the case $i = 1$, just replace $\operatorname{Hom}(C_i^2, C_{i+1})$ with $\operatorname{rad}(C_i^2, C_{i+1})$, and the argument goes through.

The dual argument, using the fact that d-almost split sequences are source, yields the claim for rows. □

4.3. Tensor products

The main tool which allows us to perform homological computations for tensor products is the Künneth formula over a field ([2, VI.3.3.1]):

Lemma 4.9. *If $X_\bullet, Y_\bullet$ are complexes, then there is a functorial isomorphism*

$$H_i(X_\bullet \otimes Y_\bullet) \cong \bigoplus_{p+q=i} H_p(X_\bullet) \otimes H_q(Y_\bullet).$$

Since tensor products of projective resolutions are projective resolutions, we immediately get

Lemma 4.10. *If $M_1, M_2 \in \operatorname{mod} A$ and $N_1, N_2 \in \operatorname{mod} B$, then there is a functorial isomorphism*

$$\operatorname{Ext}^i_{A\otimes B}(M_1 \otimes N_1, M_2 \otimes N_2) \cong \bigoplus_{p+q=i} \operatorname{Ext}^p_A(M_1, M_2) \otimes \operatorname{Ext}^q_B(N_1, N_2).$$

The total tensor product of complexes is a functor in a natural way, so we can speak of tensor products of maps of complexes (for a very general treatment of how this is done, see [2, IV.4 and IV.5]). An important result which is proved in [16] for d-representation finite algebras is also true for d-complete algebras, namely:

Theorem 4.11. *Let A, B be n- respectively m-complete algebras. Let $\operatorname{Cone}\varphi$ and $\operatorname{Cone}\psi$ be n- respectively m-almost split sequences starting in $\operatorname{add}\tau_n^i DA$ respectively $\operatorname{add}\tau_m^i DB$ for some common $i > 0$. Then $\operatorname{Cone}(\varphi \otimes \psi)$ is an $(n+m)$-almost split sequence in $\mathcal{M}(A \otimes B)$.*

Proof. This is proved in the same way as in [16, Section 3.3]. For convenience, we present the main points of the proof. By definition $\operatorname{Cone}(\varphi \otimes \psi)$ is a complex bounded between 0 and $n + m + 1$, it is exact by the Künneth formula, and it is easy to check that all maps appearing are radical. Now $\varphi : A^0_\bullet \to A^1_\bullet$ and $\psi : B^0_\bullet \to B^1_\bullet$, and by assumption we have that $A^0_j \in \operatorname{add}\tau_n^i DA$, $A^1_j \in \operatorname{add}\tau_n^{i-1} DA$, $B^0_j \in \operatorname{add}\tau_m^i DB$ and

$B_j^1 \in \operatorname{add} \tau_m^{i-1} DB$ for every j since $A_j \otimes B_j \in \mathcal{M}(A \otimes B)$. Let now $M \otimes N$ be any indecomposable in $\mathcal{M}(A \otimes B)$. We need to prove that $F_{M\otimes N}(\mathrm{Cone}(\varphi \otimes \psi))$ is exact. As in [16, Section 2.3], for a radical map of radical complexes $\eta : A_\bullet \to B_\bullet$ and a module X we can define $\tilde{F}_X(\eta) = \eta \circ - : \mathrm{Hom}(X, A_\bullet) \to F_X(B_\bullet)$. Then in our setting there is a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}(M, A_\bullet^0) \otimes \mathrm{Hom}(N, B_\bullet^0) & \xrightarrow{\cong} & \mathrm{Hom}(M \otimes N, A_\bullet^0 \otimes B_\bullet^0) \\
{\scriptstyle \tilde{F}_M(\varphi) \otimes \tilde{F}_N(\psi)} \downarrow & & \downarrow {\scriptstyle \tilde{F}_{M\otimes N}(\varphi \otimes \psi)} \\
F_M(A_\bullet^1) \otimes F_N(B_\bullet^1) & \longrightarrow & F_{M\otimes N}(A_\bullet^1 \otimes B_\bullet^1).
\end{array}$$

Now $F_{M\otimes N}(\mathrm{Cone}(\varphi \otimes \psi))$ is exact if and only if $\tilde{F}_{M\otimes N}(\varphi \otimes \psi)$ is a quasi-isomorphism. The left map in the diagram $\tilde{F}_M(\varphi) \otimes \tilde{F}_N(\psi)$ is a quasi-isomorphism since $\mathrm{Cone}(\varphi)$ and $\mathrm{Cone}(\psi)$ are n- respectively m-almost split sequences. Then it is enough to prove that the bottom map is a quasi-isomorphism, and this is done by showing that its cokernel is isomorphic to

$$F_M(A_\bullet^1) \otimes \mathrm{top}(N, B_0^1) \oplus \mathrm{top}(M, A_0^1) \otimes F_N(B_\bullet^1)$$

and then by easy verification that the above cokernel is exact. The computation of the cokernel is done explicitly in [16, Section 3.3, pp. 660–662]. □

Corollary 4.12. *Let A, B be n- respectively m-complete algebras. Then $\mathcal{M}(A \otimes B)$ has $(n+m)$-almost split sequences.*

Notice that the above theorem does not require the algebra $A \otimes B$ to be $(n+m)$-representation finite (in which case we know a priori that $(n+m)$-almost split sequences must exist). In the setting of [16], this result is about describing the structure of such sequences. In the setting of d-complete algebras, this result is used to prove that $(n+m)$-almost split sequences exist, whereas it is a priori not clear that they should.

One can also say something about injective modules (which are not the starting point of any d-almost split sequence).

Proposition 4.13. *Let A, B be n- respectively m-complete algebras, and let $\Lambda = A \otimes B$. Then for every injective Λ-module $X \otimes Y$ there is a source sequence*

$$X \otimes Y \to E_{n+m} \to \cdots \to E_1 \to 0$$

in $\mathcal{M}(\Lambda)$.

Proof. Since X and Y are injective, we have sequences in $\mathcal{M}(A)$ respectively $\mathcal{M}(B)$

$$X_\bullet = X \to C_n \to \cdots \to C_1 \to 0$$
$$Y_\bullet = Y \to D_m \to \cdots \to D_1 \to 0$$

such that

$$0 \to \mathrm{Hom}(C_1, M) \to \cdots \to \mathrm{Hom}(X, M) \to \mathrm{top}(X, M) \to 0,$$
$$0 \to \mathrm{Hom}(D_1, N) \to \cdots \to \mathrm{Hom}(Y, N) \to \mathrm{top}(Y, N) \to 0$$

are exact for all indecomposables M, N. Now consider the homology of $X_\bullet \otimes Y_\bullet$.

$$H_i(X_\bullet \otimes Y_\bullet) = \bigoplus_{p+q=i} H_p(X_\bullet) \otimes H_q(Y_\bullet) = \begin{cases} H_0(X_\bullet) \otimes H_0(Y_\bullet) \text{ if } i = n+m+2 \\ 0 \text{ else.} \end{cases}$$

So we have at least an exact sequence

$$X_\bullet \otimes Y_\bullet = X \otimes Y \to \cdots \to C_1 \otimes D_1 \to 0.$$

Apply $\mathrm{Hom}(-, M \otimes N)$ to this sequence and compute homology.

$$\begin{aligned} H_i(\mathrm{Hom}(X_\bullet \otimes Y_\bullet, M \otimes N)) &= H_i\left(\mathrm{Hom}(X_\bullet, M) \otimes \mathrm{Hom}(Y_\bullet, M)\right) = \\ &= \bigoplus_{p+q=i} H_p(\mathrm{Hom}(X_\bullet, M)) \otimes H_q(\mathrm{Hom}(Y_\bullet, M)) = \\ &= \begin{cases} \mathrm{top}(X, M) \otimes \mathrm{top}(Y, N) \text{ if } i = 0 \\ 0 \text{ else.} \end{cases} \end{aligned}$$

We will be done if we prove that $X_\bullet \otimes Y_\bullet$ is source, which amounts now to prove that

$$\mathrm{top}(X \otimes Y, M \otimes N) = H_0(\mathrm{Hom}(X_\bullet \otimes Y_\bullet, M \otimes N)) = \mathrm{top}(X, M) \otimes \mathrm{top}(Y, N).$$

By tensoring the complexes

$$0 \to \mathrm{rad}(X, M) \to \mathrm{Hom}(X, M)$$

and

$$0 \to \mathrm{rad}(Y, N) \to \mathrm{Hom}(Y, N)$$

and looking at homology, one finds an exact sequence

$$\begin{aligned} &0 \to \mathrm{rad}(X, M) \otimes \mathrm{Hom}(Y, N) + \mathrm{Hom}(X, M) \otimes \mathrm{rad}(Y, N) \to \\ &\to \mathrm{Hom}(X, M) \otimes \mathrm{Hom}(Y, N) \to \mathrm{top}(X, M) \otimes \mathrm{top}(Y, N) \to 0. \end{aligned}$$

Now the middle term is isomorphic to $\mathrm{Hom}(X \otimes Y, M \otimes N)$, and this isomorphism induces an isomorphism between the first term and $\mathrm{rad}(X \otimes Y, M \otimes N)$, hence by looking at cokernels we get

$$\begin{aligned} \mathrm{top}(X \otimes Y, M \otimes N) &\cong \frac{\mathrm{Hom}(X \otimes Y, M \otimes N)}{\mathrm{rad}(X \otimes Y, M \otimes N)} \\ &\cong \frac{\mathrm{Hom}(X, M) \otimes \mathrm{Hom}(Y, N)}{\mathrm{rad}(X, M) \otimes \mathrm{Hom}(Y, N) + \mathrm{Hom}(X, M) \otimes \mathrm{rad}(Y, N)} \\ &\cong \mathrm{top}(X, M) \otimes \mathrm{top}(Y, N) \end{aligned}$$

and we are done. □

Lemma 4.14. *Let A, B be n- respectively m-complete algebras. Then the following are equivalent:*

(1) $T_{A \otimes B} \cong T_A \otimes T_B$.

(2) $\exists l \in \mathbb{N}$ *such that* A *and* B *are* l*-homogeneous.*

Proof. (2) $\Rightarrow$ (1) is clear by definition.

To prove (1) $\Rightarrow$ (2), assume it does not hold, that is $T_{A\otimes B} \cong T_A \otimes T_B$ but there are i, j such that $l_i \neq l_j$ for the corresponding indecomposable injectives $E_i \in \operatorname{add} DA$ and $F_j \in \operatorname{add} DB$. We can assume that $l_i > l_j$, otherwise the proof is similar. Call $X_{ij} = \tau_n^{l_i-1} E_i \otimes \tau_m^{l_j-1} F_j \in \operatorname{add} T_{A\otimes B}$. Then

$$\tau_{m+n}^{-l_j+1}(X_{ij}) = \tau_n^{l_i-l_j} E_i \otimes F_j$$

is not injective, since by assumption $\tau_n^{l_i-l_j} E_i$ is not injective. On the other hand, modules in $\mathcal{M}(A \otimes B)$ which satisfy $\tau_{m+n} X = 0$ are precisely the injective $A \otimes B$-modules, and so $\tau_{m+n}^{-l_j+1}(X_{ij})$ is not in $\mathcal{M}$, contradiction. $\square$

4.4. Acyclicity

We collect here some lemmas about acyclicity which we will use.

Lemma 4.15. *The module* Λ_Λ *is directed if and only if the module* $D_\Lambda \Lambda$ *is directed.*

Proof. The Nakayama functor induces an equivalence $\nu : \operatorname{add} \Lambda_\Lambda \to \operatorname{add} D_\Lambda \Lambda$, and the definition of directedness is invariant under equivalence. $\square$

Lemma 4.16. *Let* Λ *be* d*-complete. Then* Λ *is acyclic if and only if* $\mathcal{M}$ *is directed.*

Proof. If $\mathcal{M}$ is directed, then so is $\operatorname{add} D\Lambda \subseteq \mathcal{M}$. By Lemma 4.15, Λ is then acyclic.

Conversely, if Λ is acyclic then $\operatorname{add} D\Lambda$ is directed by Lemma 4.15, and then so is $\operatorname{add} \tau_d^i D\Lambda$ for any i by Lemma 4.3. Any nonzero map between indecomposables in $\mathcal{M}$ is either within a slice $\mathcal{S}(i) = \operatorname{add} \tau_d^i D\Lambda$ or from $\mathcal{S}(i)$ to $\mathcal{S}(j)$ with $j < i$. Therefore there can be no cycles within a slice nor cycles that contain modules from different slices and $\mathcal{M}$ is directed. $\square$

Acyclicity is well suited to study d-almost split sequences.

Lemma 4.17. *Let* Λ *be* d*-complete, and let*

$$0 \longrightarrow \tau_d X \longrightarrow C_d \longrightarrow \cdots \longrightarrow C_1 \longrightarrow X \longrightarrow 0$$

be a d*-almost split sequence in* $\operatorname{mod}\Lambda$*. Then for every indecomposable summand* Y *of* $\bigoplus_{i=1}^d C_i$*, we have*

$$\tau_d X < Y < X.$$

Proof. This follows directly from Lemma 4.8 and the definition of $<$. $\square$

Let us now consider acyclicity in relation to tensor products.

Lemma 4.18. *The algebras* A *and* B *are acyclic if and only if* $\Lambda = A \otimes B$ *is acyclic.*

Proof. Let us first remark that for $X, X' \in \operatorname{mod} A$ and $Y, Y' \in \operatorname{mod} B$ we have

$$\operatorname{rad}(X \otimes Y, X' \otimes Y') = \operatorname{rad}(X, X') \otimes \operatorname{Hom}(Y, Y') + \operatorname{Hom}(X, X') \otimes \operatorname{rad}(Y, Y')$$

by [16, Lemma 3.6]. Assume $X < X$ in $\operatorname{add} A$ via $X_1, \dots, X_m$. Then for an indecomposable $P \in \operatorname{add} B$ we have that $X \otimes P < X \otimes P$ via $X_1 \otimes P, \dots, X_m \otimes P$ since

$$\operatorname{rad}(X_i \otimes P, X_{i+1} \otimes P) \supseteq \operatorname{rad}(X_i, X_{i+1}) \otimes \operatorname{End}(P) \neq 0$$

for all i. Therefore if Λ is acyclic then A is acyclic. By symmetry, if Λ is acyclic then B is acyclic as well.

Let us now prove the converse implication. Assume that $X \otimes Y < X \otimes Y$ in $\operatorname{add} \Lambda$ via $X_1 \otimes Y_1, \dots, X_m \otimes Y_m$. We can assume that $\operatorname{rad}(X, X) = 0 = \operatorname{rad}(Y, Y)$. Moreover, it cannot be that $X_i \cong X$ for all i and that $Y_j \cong Y$ for all j. Without loss of generality, assume that $X_i \ncong X$ for some i. We will prove that $X < X$ via a subsequence (Z_j) of the X_i's. We have that $\operatorname{Hom}(X_i, X_{i+1}) \neq 0$ for all i by assumption. Set $Z_0 = X$ and $Z_j = X_i$, where $i = \min\{l \mid X_l \ncong Z_{j-1}\}$ for $j > 0$. By construction, $Z_p = X$ for some p (and for $j > p$, Z_j is not defined). Then we are done, since by construction $\operatorname{Hom}(Z_i, Z_{i+1}) \neq 0$ and $Z_i \ncong Z_{i+1}$ so that $\operatorname{rad}(Z_i, Z_{i+1}) \neq 0$ since Z_i, Z_{i+1} are indecomposable. □

5. Proof of main result

From now on, let A be n-complete acyclic, let B be m-complete acyclic and let $\Lambda = A \otimes_k B$. We use the notation of Definition 2.3. There are three conditions that need to be checked to prove the main theorem (since we saw in Lemma 4.18 that Λ is acyclic), namely that properties $(A_d), (B_d), (C_d)$ in Definition 2.3 are preserved under tensor products.

Proposition 5.1. *$\operatorname{Ext}^i_\Lambda(\mathcal{M}, \mathcal{M}) = 0$ for $0 < i < n + m$.*

Proof. Let $X \otimes Y \in \mathcal{M}_P$. We have for $i < n + m$

$$\operatorname{Ext}^i(X \otimes Y, A \otimes B) = \bigoplus_{p+q=i} \operatorname{Ext}^p(X, A) \otimes \operatorname{Ext}^q(Y, B) = 0$$

so we conclude by [12, Proposition 2.5 (a)]. □

By the same formula, Λ satisfies condition (C_{n+m}):

Lemma 5.2. *$\operatorname{Ext}^i(\mathcal{M}_P, \Lambda) = 0$ for all $0 < i < n + m$.*

Proof. Use the same formula as in Proposition 5.1. □

Notice that since $\tau_{n+m} = \tau_n \otimes \tau_m$ on $\mathcal{M}$, for sufficiently big l we have $\tau^l_{n+m} D\Lambda = 0$, so $\mathcal{M}$ has an additive generator.

We now start proving that condition (A_{n+m}) holds.

For $S = S_1 \oplus S_2$ with $S_1 \in \operatorname{add} T$ and $S_2 \in \mathcal{M}_P$, define $ES = S_1 \oplus \tau_{n+m} S_2$. Note that $E^l D\Lambda = T$ for $l \gg 0$. Now fix $S = E^i D\Lambda$ for some $i \geq 0$. To check condition (A_{n+m}) for Λ, we need some preliminaries.

Lemma 5.3. *If $\operatorname{Ext}^i(S, S) = 0$ for all $i \neq 0$, then $\operatorname{Ext}^i(ES, ES) = 0$ for all $i \neq 0$.*

Proof. Since $\operatorname{Ext}^i_\Lambda(\mathcal{M}, \mathcal{M}) = 0$ for $0 < i < n + m$, it suffices to check that $\operatorname{Ext}^{n+m}(ES, ES) = 0$. Since $ES = S_1 \oplus \tau_{n+m} S_2$, consider first $M_1 \otimes N_1 \in \operatorname{add} S_1$ and $M_2 \otimes N_2 \in \operatorname{add} ES$. Then

$$\operatorname{Ext}^{n+m}(M_1 \otimes N_1, M_2 \otimes N_2) = \operatorname{Ext}^n(M_1, M_2) \otimes \operatorname{Ext}^m(N_1, N_2) = 0$$

since $M_1 \otimes N_1 \in \operatorname{add} S_1 \subseteq \operatorname{add} T$ implies that either M_1 or N_1 is relative projective in $T_A^\perp$ respectively $T_B^\perp$. This proves that $\operatorname{Ext}^{n+m}(S_1, ES) = 0$. Now let Y be an indecomposable summand of ES, and consider $\operatorname{Ext}^{n+m}(\tau_{n+m}S_2, Y)$. If Y is injective, then this is 0. Otherwise, $Y = \tau_{n+m}\tau_{n+m}^- Y$ and

$$\operatorname{Ext}^{n+m}(\tau_{n+m}S_2, Y) = \operatorname{Ext}^{n+m}(S_2, \tau_{n+m}^- Y) = 0$$

by the assumption. □

Lemma 5.4. *If S is tilting then* $\operatorname{thick} ES = \mathcal{D}^b(\Lambda)$.

Proof. Set $\mathcal{S} = \operatorname{add} S$. For $X \in \operatorname{ind}\mathcal{S}$, define $h(X)$ to be the height of X with respect to the partial order introduced in Section 4.4 on $\operatorname{ind}\mathcal{S}$ (here it is crucial that Λ be acyclic, which follows from the assumptions on A and B and Lemma 4.18), that is

$$h(X) = \max\{n \mid \exists Y_0 < \cdots < Y_n = X,\ Y_i \in \operatorname{ind}\mathcal{S}\}.$$

Notice that $X > Y$ implies $h(X) > h(Y)$, and the reverse implication holds provided that X and Y are comparable. Call $\mathcal{C}_i = \operatorname{add}(\{ES\} \cup \{Y \in \operatorname{ind}\mathcal{S} \mid h(Y) < i\})$. For $X \in \operatorname{ind}\mathcal{S}$, if $\tau_{n+m}X = 0$ then $X \in \operatorname{add} ES$. Otherwise, there is an $(n+m)$-almost split sequence

$$0 \to \tau_{n+m}X \to \cdots \to X \to 0$$

whose middle terms are in $\operatorname{add}(\{ES\} \cup \{Y \in \operatorname{ind}\mathcal{S} \mid Y < X\})$ by Lemma 4.17. In particular if $h(X) \leq i$ then the middle terms in the sequence are in

$$\operatorname{add}(\{ES\} \cup \{Y \in \operatorname{ind}\mathcal{S} \mid h(Y) < i\}) = \mathcal{C}_i.$$

It follows that $\operatorname{thick}\mathcal{C}_{i+1} \subseteq \operatorname{thick}\mathcal{C}_i$, so $\operatorname{thick}\mathcal{C}_j \subseteq \operatorname{thick}\mathcal{C}_0$ for every j. Now $\mathcal{C}_0 = \operatorname{add} ES$, and $\mathcal{C}_j = \operatorname{add}(ES \oplus S)$ for some j, so we get that $\operatorname{thick} ES = \operatorname{thick}\mathcal{C}_0 = \operatorname{thick}\mathcal{C}_j = \mathcal{D}^b(\Lambda)$ as claimed. □

Theorem 5.5. $T = T_{A\otimes B}$ *is tilting.*

Proof. By Lemma 5.3 and Lemma 5.4, if $S = E^i D\Lambda$ is tilting then $ES = E^{i+1}D\Lambda$ is tilting. Since $D\Lambda$ is tilting, and $T = E^l D\Lambda$ for some l, it follows that T is tilting. □

Now we start proving that condition (B_{n+m}) holds. We will use the following result (this is [12, Theorem 2.2(b)]):

Theorem 5.6. *Let Λ be a finite-dimensional k-algebra, $d \geq 1$ and $T \in \operatorname{mod}\Lambda$ a tilting module with* $\operatorname{proj.dim} T \leq d$. *Let $\mathcal{C} = \operatorname{add} C$ be a subcategory of $T^\perp$ such that $\operatorname{Ext}^i_\Lambda(\mathcal{C}, \mathcal{C}) = 0$ for $0 < i < d$ and $T \oplus D\Lambda \in \mathcal{C}$. Then the following are equivalent:*

(1) *$\mathcal{C}$ is a d-cluster tilting subcategory in $T^\perp$.*
(2) *Every indecomposable $X \in \mathcal{C}$ has a source sequence of the form*

$$X \to C_d \to \cdots \to C_0 \to 0$$

with $C_i \in \mathcal{C}$ for all i.

We want to apply this to $\Lambda = A \otimes B$, $\mathcal{C} = \mathcal{M}$, $T = T_{A\otimes B}$ and $d = n + m$.

Lemma 5.7. $\mathcal{M} \subseteq T^{\perp}$.

Proof. By Proposition 5.1, it is enough to check that $\operatorname{Ext}^{n+m}(T, \mathcal{M}) = 0$. Let $M_1 \otimes M_2 \in \operatorname{add} T$. Then either M_1 or M_2 is relative projective in $T_A^{\perp}$ respectively $T_B^{\perp}$, so

$$\operatorname{Ext}^{n+m}(M_1 \otimes M_2, N_1 \otimes N_2) = \operatorname{Ext}^{n}(M_1, N_1) \otimes \operatorname{Ext}^{m}(M_2, N_2) = 0$$

for any $N_1 \otimes N_2 \in \mathcal{M}$. $\square$

Theorem 5.8. *$\mathcal{M}$ is an $(n+m)$-cluster tilting subcategory of $T^{\perp}$.*

Proof. By Proposition 5.1 and Lemma 5.7, we can take $\Lambda = A \otimes B$, $\mathcal{C} = \mathcal{M}$, $T = T_{A\otimes B}$ and $d = n + m$ in the assumptions of Theorem 5.6. By Corollary 4.12 and Proposition 4.13, condition (2) is satisfied. Our claim is then the equivalent statement (1). $\square$

Now we have established everything we need to prove the main result.

Proof of Theorem 3.1. By Theorem 5.5, Theorem 5.8, and Lemma 5.2, we have that $A \otimes B$ satisfies the conditions $(A_{n+m}), (B_{n+m}), (C_{n+m})$ in the definition of $(n+m)$-complete algebra. By Lemma 4.18, $A \otimes B$ is acyclic. $\square$

Proof of Corollary 3.2. By Theorem 3.1, $A \otimes B$ is $(n+m)$-complete. By [12, Proposition 1.13], we have that $T_A \cong A$, $T_B \cong B$ and that $A \otimes B$ is $(n+m)$-representation finite if and only if $T_{A\otimes B} \cong A \otimes B$. By Lemma 4.14, this happens if and only if A and B are l-homogeneous for some common l. $\square$

6. Examples

Let us consider one of the simplest non-trivial examples. Let $A = B = kQ$, where Q is the quiver

$$1 \longleftarrow 2.$$

Then $\Lambda = A \otimes B$ is the quiver algebra of a commutative square. This algebra is 2-complete, since the factors are 1-representation finite. It is not 2-representation finite since the factors are not homogeneous. However, Λ is representation finite, so we can draw the entire Auslander–Reiten quiver of Λ. We represent modules by their dimension vector.

$$\begin{tikzcd}[column sep=small]
 & {\begin{smallmatrix}0&0\\1&1\end{smallmatrix}} \arrow[dr] & & {\begin{smallmatrix}0&1\\0&0\end{smallmatrix}} \arrow[dr] & & {\begin{smallmatrix}1&0\\1&0\end{smallmatrix}} \arrow[dr] & \\
{\begin{smallmatrix}0&0\\1&0\end{smallmatrix}} \arrow[ur] \arrow[dr] & & {\begin{smallmatrix}1&0\\1&1\end{smallmatrix}} \arrow[ur] \arrow[r] \arrow[dr] & {\begin{smallmatrix}1&1\\1&1\end{smallmatrix}} \arrow[r] & {\begin{smallmatrix}1&1\\0&1\end{smallmatrix}} \arrow[ur] \arrow[dr] & & {\begin{smallmatrix}0&1\\0&0\end{smallmatrix}} \\
 & {\begin{smallmatrix}1&0\\1&0\end{smallmatrix}} \arrow[ur] & & {\begin{smallmatrix}0&0\\0&1\end{smallmatrix}} \arrow[ur] & & {\begin{smallmatrix}1&1\\0&0\end{smallmatrix}} \arrow[ur] &
\end{tikzcd}$$

In this case,

$$T = {\begin{smallmatrix}0&0\\1&0\end{smallmatrix}} \oplus {\begin{smallmatrix}1&1\\1&1\end{smallmatrix}} \oplus {\begin{smallmatrix}0&1\\0&1\end{smallmatrix}} \oplus {\begin{smallmatrix}1&1\\0&0\end{smallmatrix}}$$

and

$$\mathcal{M} = \operatorname{add} M = \operatorname{add}(T \oplus {\scriptstyle\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}}).$$

One can explicitly compute all Ext-groups of all pairs of indecomposables, since we have only finitely many. If we represent by $\otimes$ the indecomposables in $\operatorname{add} T$, by $\odot$ the ones in $\mathcal{M}$ but not in $\operatorname{add} T$, by $\blacksquare$ the ones in $T^{\perp}$ but not in $\mathcal{M}$, and by $\cdot$ the ones outside $T^{\perp}$, we get the following picture:

$$\begin{tikzcd}
 & \cdot \arrow[dr] & & \cdot \arrow[dr] & & \otimes \arrow[dr] & \\
\otimes \arrow[ur] \arrow[dr] & & \blacksquare \arrow[ur] \arrow[r] \arrow[dr] & \otimes \arrow[r] & \blacksquare \arrow[ur] \arrow[dr] & & \odot \\
 & \cdot \arrow[ur] & & \cdot \arrow[ur] & & \otimes \arrow[ur] &
\end{tikzcd}$$

It can be checked that both the indecomposable modules in $T^{\perp} \setminus \mathcal{M}$ have extensions with M on both sides, as it is required by the definition of 2-cluster tilting. Here we find that $\mathcal{M}$ is 2-cluster tilting in $T^{\perp}$.

The Auslander–Reiten quiver of $\operatorname{add}(M)$ is given by

$$\begin{tikzcd}
 & & \begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix} \arrow[dr] & \\
\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix} \arrow[r] & \begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \\
 & & \begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix} \arrow[ur] &
\end{tikzcd}$$

and this is also a picture of the only 2-almost split sequence we have.

As a second example, consider the quiver Q':

$$\begin{tikzcd}
 & & 2 \arrow[dl] \\
3 \arrow[r] & 1 & \\
 & & 4 \arrow[ul]
\end{tikzcd}$$

and the corresponding path algebra $A' = kQ'$. The Auslander–Reiten quiver of A' looks like

$$\begin{tikzcd}
 & P_2 \arrow[dr] & & N_2 \arrow[dr] & & I_2 \\
P_1 \arrow[ur] \arrow[r] \arrow[dr] & P_3 \arrow[r] & N_1 \arrow[ur] \arrow[r] \arrow[dr] & N_3 \arrow[r] & I_1 \arrow[ur] \arrow[r] \arrow[dr] & I_3 \\
 & P_4 \arrow[ur] & & N_4 \arrow[ur] & & I_4
\end{tikzcd}$$

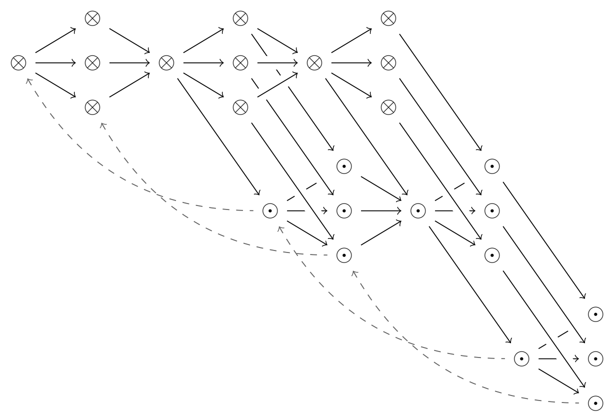
We take $B' = kQ''$, where Q'' is the quiver

$$a \longleftarrow b \longleftarrow c$$

The Auslander–Reiten quiver of B' looks like

$$\begin{tikzcd}
 & & P_c = I_a \arrow[dr] & & \\
 & P_b \arrow[ur] \arrow[dr] & & I_b \arrow[dr] & \\
P_a \arrow[ur] & & S_b \arrow[ur] & & I_c
\end{tikzcd}$$

These algebras are both 1-representation finite, so in particular they are 1-complete. Their tensor product $\Lambda' = A' \otimes B'$ is therefore 2-complete. It is not 2-representation finite since B' is not homogeneous. In this example, we cannot draw the entire module category of Λ', but we still have complete control over the "higher Auslander–Reiten quiver" of Λ', that is the Auslander–Reiten quiver of $\mathrm{add}(M)$:



Here the dashed arrows represent τ_2, and we have drawn them only between some modules to avoid clogging the picture. We have again written $\otimes$ for indecomposable summands of T, and $\odot$ for the other indecomposable summands of M. It should be clear from the picture which module corresponds to which node.

Notice that this example presents some regularity which is not to be expected in general, since we have taken A' to be homogeneous. Moreover, in this example (and in general) we cannot directly check that arbitrary modules in $\mathrm{mod}\,\Lambda'$ which are in $T^{\perp}$ have extensions on both sides with $\mathcal{M}$.

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Paper III

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Self-Injective Jacobian Algebras from Postnikov Diagrams

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Abstract
We study a finite-dimensional algebra Λ from a Postnikov diagram D in a disk, obtained from the dimer algebra of Baur-King-Marsh by factoring out the ideal generated by the boundary idempotent. Thus, Λ is isomorphic to the stable endomorphism algebra of a cluster tilting module $T \in \mathrm{CM}(B)$ introduced by Jensen-King-Su in order to categorify the cluster algebra structure of $\mathbb{C}[\mathrm{Gr}_k(\mathbb{C}^n)]$. We show that Λ is self-injective if and only if D has a certain rotational symmetry. In this case, Λ is the Jacobian algebra of a self-injective quiver with potential, which implies that its truncated Jacobian algebras in the sense of Herschend-Iyama are 2-representation finite. We study cuts and mutations of such quivers with potential leading to some new 2-representation finite algebras.

Keywords Dimer model · Postnikov diagram · Self-injective algebra · Jacobian algebra · Preprojective algebra · Higher dimensional Auslander-Reiten theory · Grassmannian cluster algebra

1 Introduction

In this article we study algebras constructed from (k, n)-Postnikov diagrams. These are configurations of oriented curves in the disk satisfying some axioms, and were defined in [15] to study total positivity of the Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$. The combinatorial data of such a diagram has been shown in [14] to be equivalent to the data of a maximal noncrossing collection of k-element subsets of $\{1, \ldots, n\}$.

To a Postnikov diagram D one can associate (see [4]) a planar ice quiver with potential $(Q, W, F) = (Q, W, F)(D)$, and consider the frozen Jacobian algebra $A = A(D)$ (which is infinite dimensional). If one then quotients out the idempotent e corresponding to the

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boundary (frozen) vertices, one gets a quiver with potential $(\underline{Q}, \underline{W})$ whose Jacobian algebra $\Lambda = \Lambda(D)$ is finite dimensional, and this is the main object of our study.

One can also label the vertices of $Q(D)$ by the k-element subsets appearing in the maximal noncrossing collection corresponding to D. Postnikov diagrams were used in [18] to show that the homogeneous coordinate ring of $\mathrm{Gr}_k(\mathbb{C}^n)$ is a cluster algebra: the k-element subsets are labels for the Plücker coordinates, which are cluster variables. The maximal noncrossing collections correspond precisely to clusters, and indeed the quiver Q corresponds to the quiver of the cluster given by its collection. By [14], every cluster consisting of Plücker coordinates appears in this way (since all maximal noncrossing collections appear as the labels of such a quiver $Q(D)$).

There is an algebra B, depending only on k and n, which was used in [12] to categorify the cluster algebra structure of the homogeneous coordinate ring of $\mathrm{Gr}_k(\mathbb{C}^n)$, building on the categorification of the coordinate ring of an affine open cell constructed in [7]. The categorification takes place in $\mathrm{CM}(B)$, where Jensen-King-Su define a Cohen-Macaulay B-module L_I of rank 1 for every k-element subset I of $\{1, \dots n\}$. Given a maximal noncrossing collection $\mathbb{I}$, one can define a module

$$T = \bigoplus_{I \in \mathbb{I}} L_I$$

and this module is shown in [12] to be cluster tilting in $\mathrm{CM}(B)$. One of the main results in [4] is that there is an isomorphism

$$A \cong \mathrm{End}_B(T)$$

where A is the frozen Jacobian algebra corresponding to the Postnikov diagram associated to $\mathbb{I}$. The frozen vertices correspond to projective-injective B-modules, so there is an isomorphism

$$\Lambda \cong \underline{\mathrm{End}}_B(T).$$

It turns out that Λ is the same if we take the completed algebra $\hat{B}$ instead of B, so we can use results about the completed case. The stable category $\underline{\mathrm{CM}}(\hat{B})$ is a 2-Calabi-Yau triangulated category with a cluster tilting object T, and we can use the machinery of [10] to prove results about the algebra $\underline{\mathrm{End}}_{\hat{B}}(T)$ (which is, as we said, isomorphic to Λ).

Our main result is that Λ is self-injective if and only if D is symmetric under rotation by $2k\pi/n$ (which corresponds to $\mathbb{I}$ being invariant with respect to adding k to all elements).

Thus Postnikov diagrams turn out to be a new source of planar self-injective quivers with potential in the sense of [9]. Previously, the only known planar self-injective quivers with potential were mutation equivalent to so called "squares", "triangles", or "n-gons", in the terminology of [9, §9]. The algebras coming in this way from "n-gons" are precisely the self-injective cluster tilted algebras classified by Ringel [17]. We construct some new examples not belonging to the above families, thus answering [9, Question 10.1(1)] in the negative. In fact, two counterexamples had already been found (and we recover them), but they were not published. In particular, we construct an infinite family of algebras for which the Nakayama permutation has arbitrarily large order. Previously, the only known self-injective planar quivers with potential with Nakayama permutation of order at least 6 were mutation equivalent to "n-gons", and our examples are not of this type.

Self-injective Jacobian algebras are precisely the 3-preprojective algebras of 2-representation finite algebras [9]. The latter can be constructed by choosing an appropriate set of arrows C (called a cut) in the quiver. We exploit the results of Herschend-Iyama to prove that for a given symmetric Postnikov diagram, all such 2-representation finite algebras Λ_C are iterated 2-APR tilts of each other, so in particular they are derived equivalent.

Moreover, it is interesting to know when a cut is invariant under the Nakayama automorphism, since in this case Λ_C is twisted $2\frac{l-1}{l}$-Calabi-Yau for some l [8]. In our setting, the Nakayama automorphism is simply given by rotation in the plane, so this condition is easily accessible.

One can study mutations of cluster tilting objects, of quivers with potential, or of Postnikov diagrams (the latter is called geometric exchange). These all correspond to each other, with the caveat that only certain vertices of the quiver become mutable (since geometric exchange only works for some regions of the disk). In [15] it is proved that geometric exchange is transitive on the set of (k, n)-Postnikov diagrams, and we deduce that mutation is transitive on cluster tilting objects in $\mathrm{CM}(\hat{B})$ whose indecomposable summands have rank 1. We also give a direct proof of a special case of a theorem which appeared in [9] about mutations along a Nakayama orbit.

It should be noted that many of the statements we present are combinations of published results and probably known to experts, even though they cannot be found in the literature as we state them. The original contributions of this paper are in the results of Section 8, Section 9 and in the new examples of Section 10.

The structure of this article is as follows. In Section 2 we set up some notation and conventions. In Section 3 we recall the definitions we need about ice quivers with potential and frozen Jacobian algebras. In Section 4 we define Postnikov diagrams, explain their combinatorics and use them to construct ice quivers with potential. In Section 5 we collect some results about cluster tilting objects with self-injective endomorphism algebras. In Section 6 we define the algebra B and the modules L_I, as well as compute the action of the Serre functor of $\underline{\mathrm{CM}}(\hat{B})$ on the modules L_I. In Section 7 we define the module T and study cluster tilting objects and their mutations in $\mathrm{CM}(\hat{B})$. We interpret those mutations in terms of mutations of quivers with potential and geometric exchange. In Section 8 we consider Postnikov diagrams which are rotation symmetric, and prove our main result. In Section 9 we study cuts for self-injective quivers with potential arising from symmetric Postnikov diagrams. In Section 10 we present some examples of self-injective quivers with potential constructed in this way. We recover an infinite family found in [9], as well as some members of another infinite family. We construct a new infinite family, and finally some sporadic cases.

2 Notation and Conventions

By an algebra Λ we mean a unital, associative and basic $\mathbb{C}$-algebra unless otherwise specified. We write $\Lambda \operatorname{mod}$ ($\operatorname{mod} \Lambda$) for the category of finitely generated left (right) Λ-modules. If Λ is graded by an abelian group G, we write $\Lambda \operatorname{mod}^G$ ($\operatorname{mod}^G \Lambda$) for the category of G-graded finitely generated left (right) Λ-modules. In various contexts we will denote by D the functor $\operatorname{Hom}_{\mathbb{C}}(-, \mathbb{C})$. Unless otherwise specified, "module" means object of $\Lambda \operatorname{mod}$. If $\varphi : \Lambda \to \Lambda$ is a ring automorphism and $M \in \Lambda \operatorname{mod}$, we define ${}_{\varphi}M \in \Lambda \operatorname{mod}$ to be M as an abelian group, with $a *_{\varphi} m = \varphi(a)m$. Similarly we define M_{φ} by $m *_{\varphi} a = m\varphi(a)$, for $M \in \operatorname{mod} \Lambda$. The composition $g \circ f$ means that f is applied first and g second.

Throughout this article, we will fix two positive integers $k \leq n$. We denote by $[n]$ the set $\mathbb{Z}/n\mathbb{Z}$, usually equipped with the cyclic ordering. We write $\binom{[n]}{k}$ for the set of k-element subsets of $[n]$. For a subset I of $[n]$, we write

$$I + k := \{i + k \mid i \in I\} \subseteq [n]$$

and for a subset $\mathbb{I}$ of $\binom{[n]}{k}$, we write

$$\mathbb{I}+k := \{I+k \mid I \in \mathbb{I}\} \subseteq \binom{[n]}{k}.$$

3 Ice Quivers with Potential

In this section we recall some definitions, notation and facts about (ice) quivers with potential (see [3] for a reference). Let $Q = (Q_0, Q_1)$ be a finite quiver without loops and 2-cycles. We can complete the path algebra $\mathbb{C}Q$ with respect to the $\langle Q_1\rangle$-adic topology, and denote the completion by $\widehat{\mathbb{C}Q}$. A *potential* on Q is an element

$$W \in \widehat{\mathbb{C}Q} \Big/ \overline{\left[\widehat{\mathbb{C}Q}, \widehat{\mathbb{C}Q}\right]},$$

where $\left[\widehat{\mathbb{C}Q}, \widehat{\mathbb{C}Q}\right]$ is the vector space spanned by commutators in $\widehat{\mathbb{C}Q}$, and $\overline{}$ denotes closure in the $\langle Q_1\rangle$-adic topology. In other words, W is a (possibly infinite) linear combination of cycles in Q, where we identify cycles up to cyclic permutation of their arrows. We say that W is finite if it can be written as a finite such linear combination. For $a \in Q_1$, we can define the *cyclic derivative* $\partial_a : \widehat{\mathbb{C}Q} \to \widehat{\mathbb{C}Q}$ by

$$\partial_a(a_1 \cdots a_l) = \sum_{a_i = a} a_{i+1} \cdots a_l a_1 \cdots a_{i-1}$$

and extended by linearity and continuity on $\widehat{\mathbb{C}Q}$. We also get an induced map ∂_a : $\widehat{\mathbb{C}Q} \Big/ \overline{\left[\widehat{\mathbb{C}Q}, \widehat{\mathbb{C}Q}\right]} \to \widehat{\mathbb{C}Q}$.

Definition 3.1 A *quiver with potential* is a pair (Q, W) where Q is a quiver without loops and 2-cycles and W is a potential on Q. The *Jacobian algebra* $\hat{\wp}(Q, W)$ is the algebra

$$\hat{\wp}(Q, W) = \widehat{\mathbb{C}Q} \Big/ \overline{\langle \partial_a W \mid a \in Q_1\rangle}.$$

We can generalise this definition slightly by allowing frozen vertices.

Definition 3.2 An *ice quiver with potential* is a triple (Q, W, F) where (Q, W) is a quiver with potential, and F is a subset of Q_0 (the elements of F are called *frozen vertices*). Call Q_F the set of arrows of Q that start and end at a frozen vertex. The *frozen Jacobian algebra* $\hat{\wp}(Q, W, F)$ is the algebra

$$\hat{\wp}(Q, W, F) = \widehat{\mathbb{C}Q} \Big/ \overline{\langle \partial_a W \mid a \in Q_1 \setminus Q_F\rangle}.$$

In other words, we do not take derivatives with respect to arrows between the frozen vertices.

Given an ice quiver with potential (Q, W, F), one can construct a quiver with potential $(\underline{Q}, \underline{W})$ as follows. Set $\underline{Q}$ to be the quiver obtained from Q by removing the frozen vertices and all adjacent arrows, and define $\underline{W}$ to be the image of W under the quotient map $\widehat{\mathbb{C}Q} \to \widehat{\mathbb{C}\underline{Q}}$. Then we have $\hat{\wp}(Q, W, F)/\langle F\rangle \cong \hat{\wp}(\underline{Q}, \underline{W})$, where $\langle F\rangle$ is the ideal generated by the sum of the idempotents corresponding to vertices in F.

If W is finite, we can also define a non-completed Jacobian algebra $\wp(Q, W)$ by the same construction without all the completions. In this article, the quivers with potential $(\underline{Q}, \underline{W})$ which appear have the property that the completed and non-completed Jacobian algebras are isomorphic. In the rest of this section, we lay the ground for proving this. Let

(Q, W) be a quiver with finite potential. There is a canonical map $\wp(Q, W) \to \widehat{\wp}(Q, W)$, but this map is in general neither injective nor surjective.

Proposition 3.3 *If (Q, W) is a quiver with finite potential such that $\langle \partial_a W \mid a \in Q_1 \rangle$ is an admissible ideal of $\mathbb{C}Q$, then the canonical map $\wp(Q, W) \to \widehat{\wp}(Q, W)$ is an isomorphism.*

Proof Call $I = \langle \partial_a W \mid a \in Q_1 \rangle \subseteq \mathbb{C}Q$ and $\hat{I} = \overline{\langle \partial_a W \mid a \in Q_1 \rangle} \subseteq \widehat{\mathbb{C}Q}$. Call J and $\hat{J}$ the arrow ideals of $\mathbb{C}Q$ and $\widehat{\mathbb{C}Q}$ respectively. By assumption we have that there exists $N \gg 0$ such that $J^N \subseteq I$ and then $\hat{J}^N \subseteq \hat{I}$. Observe that we have that $\widehat{\mathbb{C}Q} = \mathbb{C}Q + \hat{J}^N$, and that $J^N = \mathbb{C}Q \cap \hat{J}^N$. There is a commutative diagram

$$\begin{tikzcd}
J^N \arrow[r, hook] \arrow[d, hook] & I \arrow[r, hook] \arrow[d, hook] & \mathbb{C}Q \arrow[r, two heads] \arrow[d, hook] & \wp(Q, W) \arrow[d] \\
\hat{J}^N \arrow[r, hook] & \hat{I} \arrow[r, hook] & \widehat{\mathbb{C}Q} \arrow[r, two heads] & \widehat{\wp}(Q, W).
\end{tikzcd}$$

We get an induced commutative diagram

$$\begin{tikzcd}
I/J^N \arrow[r, hook] \arrow[d] & \mathbb{C}Q/J^N \arrow[r, two heads] \arrow[d, "\cong"] & \wp(Q, W) \arrow[d] \\
\hat{I}/\hat{J}^N \arrow[r, hook] & \widehat{\mathbb{C}Q}/\hat{J}^N \arrow[r, two heads] & \widehat{\wp}(Q, W),
\end{tikzcd}$$

so it is enough to show that the map $I/J^N \to \hat{I}/\hat{J}^N$ is an isomorphism. This map is injective since $J^N = \mathbb{C}Q \cap \hat{J}^N$. Moreover, $\hat{I} \subseteq I + \hat{J}^N$ since $I + \hat{J}^N$ is closed, so the map is surjective. □

Corollary 3.4 *Let (Q, W, F) be an ice quiver with potential. Suppose that W is finite and that every sufficiently long path is equal in $\wp(Q, W, F)$ to a path that goes through a frozen vertex. Then $\wp(\underline{Q}, \underline{W}) \cong \widehat{\wp}(\underline{Q}, \underline{W})$.*

Proof The assumption means exactly that the ideal $\left\langle \partial_a \underline{W} \mid a \in \underline{Q}_1 \right\rangle$ is admissible. □

4 Postnikov Diagrams

Let us recall the definition of a (k, n)-Postnikov diagram ([15, §14], [4, Definition 2.1]).

Definition 4.1 A (k, n)-*Postnikov diagram D* consists of n directed smooth curves (strands), in a disk with n marked points on the boundary, clockwise labelled $1, 2, \ldots, n$. The strands are also labelled, with strand i starting at i and ending at $i + k$. The following axioms must hold:

(1) All crossings are transverse crossings between two distinct strands.
(2) There are finitely many crossings.
(3) Proceeding along a given strand, the other strands crossing it alternate between crossing it from the right and from the left.
(4) If two strands cross at distinct points P_1 and P_2, then one strand is oriented from P_1 to P_2 and the other from P_2 to P_1.

For axiom (3), we consider that strands cross at the boundary vertices in the obvious way. A Postnikov diagram is defined up to isotopy that fixes the boundary. Two Postnikov diagrams are *equivalent* if they are related by a sequence of twisting and untwisting moves as shown in Fig. 1. The same moves with opposite orientations are also allowed. The moves have to be executed inside a disk with no other strand involved. A Postnikov diagram is *reduced* if no untwisting moves can be applied to it.

A Postnikov diagram divides the disk into regions, whose boundaries consist of strand segments and pieces of the boundary circle. There are three kinds of such regions, according to whether their boundary is oriented clockwise, counterclockwise, or alternating in orientation (ignoring the boundary of the disk). Each alternating region can be assigned a label $I \in \binom{[n]}{k}$ consisting of the names of the strands that have this region to their left side. These labels are all distinct. Figure 2 shows a reduced Postnikov diagram with labelled alternating regions. Not all Postnikov diagrams have rotational symmetry, but we are particularly interested in symmetric ones. We call $\mathbb{I} = \mathbb{I}(D)$ the set of labels corresponding to D.

Definition 4.2 Two sets $I, J \in \binom{[n]}{k}$ are said to be *noncrossing* or *weakly separated* (see [15, Definition 3]) if there exist no cyclically ordered $a, b, c, d \in [n]$ with $a, c \in I \setminus J$ and $b, d \in J \setminus I$. We call a collection of k-element subsets of $[n]$ a *noncrossing collection* if its elements are pairwise noncrossing. We call it a *maximal noncrossing collection* if it is maximal with respect to inclusion.

Theorem 4.3 [14, Theorem 11.1] *Maximal noncrossing collections of elements of* $\binom{[n]}{k}$ *are precisely sets of labels of alternating regions in reduced* (k, n)*-Postnikov diagrams.*

Such collections are known to have $k(n-k)+1$ elements (this was conjectured in [18] and proved in [14, Theorem 4.7]). There is an explicit construction of a Postnikov diagram having a prescribed maximal noncrossing collection as labels [14, §9], and this turns out to be unique (up to equivalence). So the datum of a Postnikov diagram D is equivalent to the datum of a maximal noncrossing collection $\mathbb{I}$.

Remark 4.4 The label of the alternating region adjacent to the boundary segment of the disk from i to $i+1$ is the set $\{i-k+1, \ldots, i\}$ for $i \in [n]$.

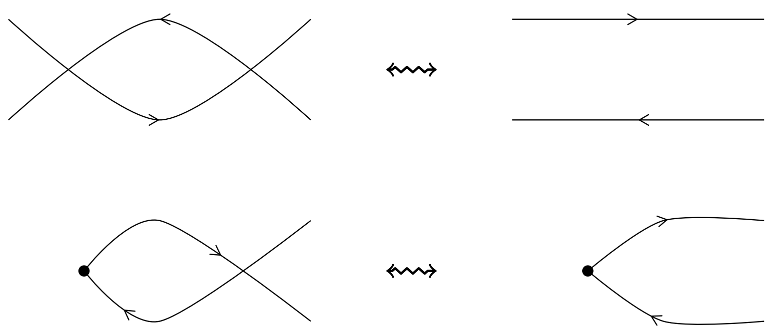

Fig. 1 Twisting and untwisting moves in a Postnikov diagram

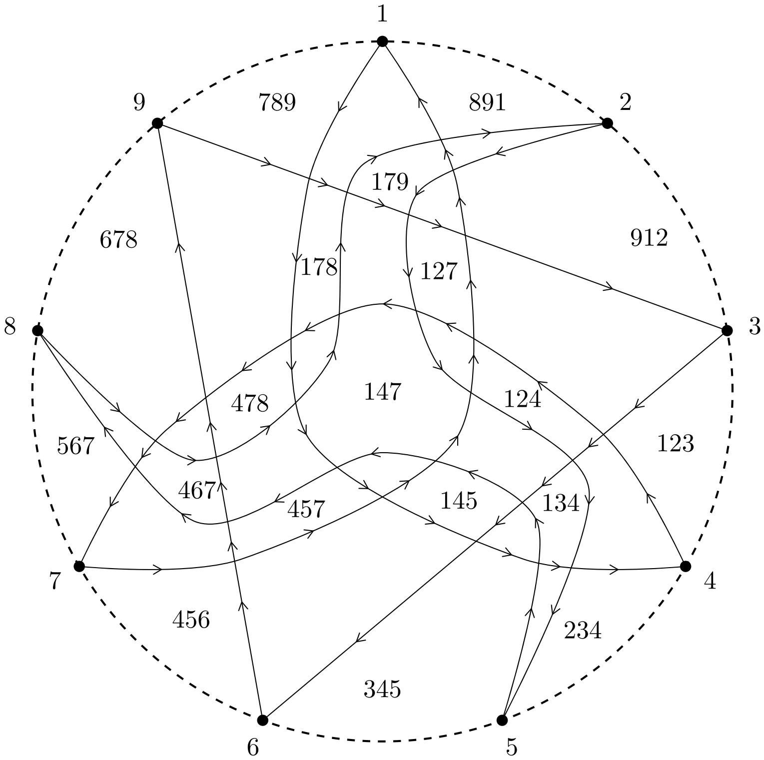

Fig. 2 A symmetric (3, 9)-Postnikov diagram

For simplicity, we assume from now on that Postnikov diagrams are reduced. To a Postnikov diagram D we can associate (see [4, §3]) an ice quiver with potential $(Q, W, F) = (Q, W, F)(D)$ such that:

(1) Vertices of Q are elements of $\mathbb{I}(D)$.
(2) Arrows of Q correspond to intersection points of alternating regions, with orientation so that the arrows "point in the same direction as the strands", as illustrated in Fig. 3.
(3) The potential W is given by the sum of cycles corresponding to the clockwise regions minus the sum of the cycles corresponding to the counterclockwise regions.
(4) The frozen vertices are the boundary vertices, i.e. the vertices corresponding to the boundary segments of the disk.

Notice that there is a natural embedding of Q in the disk. The assumption that D is reduced implies that there are no 2-cycles in Q (which we require in our definition of ice quivers with potential).

Thus we define the frozen Jacobian algebra $A = A(D) = \wp(Q, W, F)$ (this is the dimer algebra A defined in [4]). It is proved in [4, Lemma 12.1] that the algebra A is invariant up to isomorphism under equivalence of Postnikov diagrams. Call e the idempotent of A given by the sum of the idempotents corresponding to the frozen vertices of Q. Then $eAe \subseteq A$ is an idempotent subalgebra isomorphic (see Section 7) to the opposite of the algebra B we discuss in Section 6. The algebra eAe is the boundary algebra studied in [4], and the algebra B was introduced in [12]. We are especially interested in studying the algebra $\Lambda = A/AeA$.

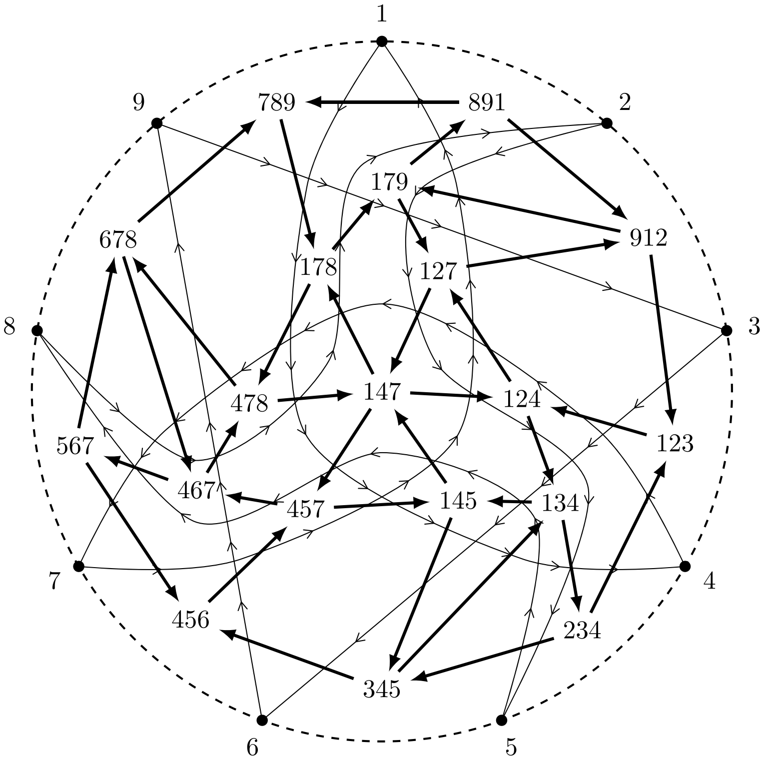

Fig. 3 The quiver associated to the Postnikov diagram in Fig. 2

The latter is the Jacobian algebra $\wp(\underline{Q}, \underline{W})$, where $\underline{Q}$ is the quiver obtained from Q by removing the frozen vertices and the adjacent arrows, and $\underline{W}$ is the image of W under the corresponding quotient map $\mathbb{C}Q \to \mathbb{C}\underline{Q}$ (see Section 3).

We are interested in the case where the Postnikov diagram D is symmetric under a rotation in the plane around the center of the disk. In particular, we consider invariance under ρ, the clockwise rotation by $2\pi k/n$. Since this notion is not invariant under isotopy, we call a Postnikov diagram *symmetric* if it is equivalent to one which is invariant under ρ. Another way of thinking about a symmetric Postnikov diagram is saying that it is equal (or isotopic) to the Postnikov diagram obtained by changing the labels of the points on the disk, replacing every i with $i + k$. In this case we have

Lemma 4.5 *Let $\mathbb{I}$ be a maximal noncrossing collection in $\binom{[n]}{k}$. Then $\mathbb{I} = \mathbb{I} + k$ if and only if there exists a symmetric Postnikov diagram D with $\mathbb{I} = \mathbb{I}(D)$.*

Proof If D is symmetric, it follows that if strand i crosses in order the strands $i_1, i_2, \dots, i_l$, then strand $i + k$ crosses in order the strands $i_1 + k, i_2 + k, \dots i_l + k$. Thus I is the label of a region of D if and only if $I + k$ is.

Conversely, assume that $\mathbb{I} = \mathbb{I}+k$. We refer to [14, §9] for the construction of a Postnikov diagram D with $\mathbb{I}(D) = \mathbb{I}$. The construction proceeds by defining a 2-dimensional CW-complex $\Sigma(\mathbb{I})$ whose vertex set is $\mathbb{I}$, embedding it in the plane, and constructing strands as zig-zag paths. It is enough to observe that the images of the vertex sets of $\Sigma(\mathbb{I})$ and of

$\Sigma(\mathbb{I}+k)$ are related by rotation in the plane. This is true since the map is as follows. One takes $v_1, \ldots, v_n$ to be the vertices of a convex n-gon in $\mathbb{R}^2$, and one maps I to $\sum_{a \in I} v_a$. We can in particular choose the n-gon to be regular and centred at the origin, and then the claim follows. $\square$

Notice that D is symmetric if and only if Q is invariant under ρ. Moreover, ρ must in this case map (counter-)clockwise cycles in Q to (counter-)clockwise cycles, so it maps W to itself, and so induces an automorphism Ψ of A. Since ρ maps F to F, this induces an automorphism of Λ which we still denote by Ψ.

We will need the following definition in Section 7.

Definition 4.6 [4] From the relations in the definition of A it follows that for any vertex $I \in Q_0$, the cyclic paths appearing in the potential and starting at I are equal to the same element in A. We denote this element by $u_I \in A$, and define

$$u = \sum_{I \in Q_0} u_I \in A.$$

Remark 4.7 It is easy to see that $u \in Z(A)$.

5 Cluster Tilting in 2-Calabi-Yau Categories

Let $\mathcal{C}$ be a $\mathbb{C}$-linear, Hom-finite triangulated category.

Definition 5.1 The category $\mathcal{C}$ is *2-Calabi-Yau* if there is a functorial isomorphism

$$\operatorname{Hom}_{\mathcal{C}}(X, Y) \cong D \operatorname{Hom}_{\mathcal{C}}(Y, X[2]).$$

An object $T \in \mathcal{C}$ is *cluster tilting* if

$$\operatorname{add} T = \{X \in \mathcal{C} \mid \operatorname{Hom}_{\mathcal{C}}(T, X[1]) = 0\}.$$

We call a cluster tilting object T *self-injective* if $\operatorname{End}_{\mathcal{C}}(T)$ is a finite-dimensional self-injective $\mathbb{C}$-algebra. For convenience, we assume cluster tilting objects to be basic.

Let us recall some facts and fix some notation about self-injective algebras. Let Λ be a finite-dimensional algebra, and let us fix a maximal set $\{e_1, \ldots, e_l\}$ of orthogonal idempotents. Then $\mathcal{P}_i = \Lambda e_i$ is a projective indecomposable Λ-module, and $\mathcal{I}_i = D(e_i \Lambda)$ is an injective indecomposable Λ-module. If $T = \bigoplus T_i$ is a basic B-module for some algebra B, with indecomposable summands T_i, and $\Lambda = \operatorname{End}_B(T)$, then we choose $\mathcal{P}_i = \operatorname{Hom}_B(T_i, T)$ and $\mathcal{I}_i = D \operatorname{Hom}_B(T, T_i)$.

An algebra Λ is self-injective if and only if there exists an automorphism $\psi : \Lambda \to \Lambda$ such that $\Lambda_\psi \cong D\Lambda$ as $\Lambda - \Lambda$-bimodules. This is called a Nakayama automorphism of Λ, and is unique up to inner automorphisms. In this case, we have that

$$\mathcal{P}_i \cong \mathcal{I}_{\sigma(i)}$$

as left Λ-modules for some permutation σ (i.e. $\Lambda e_i \cong D(e_{\sigma(i)}\Lambda)$), and

$$\Lambda_\psi \otimes_\Lambda \mathcal{P}_{\sigma(i)} \cong \mathcal{P}_i$$

for the same σ. This permutation does not depend on the choice of ψ, and is called the Nakayama permutation.

Let us now fix a 2-Calabi-Yau category $\mathcal{C}$. The following characterisation of self-injective cluster tilting objects will be useful.

Proposition 5.2 [10, Proposition 3.6], [9, Proposition 4.4] *Let $T = \bigoplus_{i=1}^{l} T_i \in \mathcal{C}$ be a cluster tilting object, with indecomposable summands T_i. Then*

(1) *T is self-injective if and only if $T \cong T[2]$.*
(2) *In this case, the permutation σ defined by $T_{\sigma(i)} \cong T_i[2]$ is the Nakayama permutation of* $\operatorname{End}_{\mathcal{C}}(T)$.

Proof Part (1) is proved in [10]. For part (2), observe

$$\mathcal{P}_i = \operatorname{Hom}(T_i, T) \cong D\operatorname{Hom}(T, T_i[2]) \cong D\operatorname{Hom}(T, T_{\sigma(i)}) = \mathcal{I}_{\sigma(i)}.$$

□

Now consider $\Lambda = \operatorname{End}_{\mathcal{C}}(T)$ as in Proposition 5.2, and fix an isomorphism $\varphi : T \to T[2]$. Then define an automorphism $\psi : \Lambda \to \Lambda$ by

$$\psi(\lambda) = \varphi[-2] \circ \lambda[-2] \circ (\varphi[-2])^{-1}.$$

Then we have

Proposition 5.3 *The map ψ is a Nakayama automorphism of Λ.*

Proof First we define a left module morphism $\Lambda \to D\Lambda$. The Serre functor [2] gives an isomorphism of bifunctors

$$\operatorname{Hom}_{\mathcal{C}}(-, ?) \cong D\operatorname{Hom}_{\mathcal{C}}(?, -[2]).$$

In our case this induces an isomorphism of vector spaces

$$\Lambda \to D\operatorname{Hom}_{\mathcal{C}}(T, T[2])$$

which we will denote by $a \mapsto a^*$ for $a \in \Lambda$. Moreover, there is an isomorphism of vector spaces

$$D\operatorname{Hom}_{\mathcal{C}}(T, T[2]) \to D\Lambda$$

given by $F \mapsto F(\varphi \circ -)$. Call $m : \Lambda \to D\Lambda$ the composition of these, i.e.

$$m : a \mapsto a^*(\varphi \circ -)$$

for all $a \in \Lambda$. Let us check that m is a left module morphism. We have $(\lambda a)^* = a^*(- \circ \lambda)$ for $\lambda, a \in \Lambda$. So

$$\lambda m(a) = a^*(\varphi \circ - \circ \lambda) = m(\lambda a).$$

Let us prove that m is a right module morphism $\Lambda_\psi \to D\Lambda$. For $a, b \in \Lambda$, we have that

$$m(ab) = (ab)^*(\varphi \circ -) = a^*(b[2] \circ \varphi \circ -).$$

The right action of Λ on $D\Lambda$ is given by $F\lambda = F(\lambda \circ -)$, so

$$(m(a))\lambda = a^*(\varphi \circ \lambda \circ -).$$

On the other hand,

$$\begin{aligned} m(a *_\psi \lambda) &= m(a \circ \varphi[-2] \circ \lambda[-2] \circ \varphi^{-1}[-2]) = \\ &= a^*((\varphi[-2] \circ \lambda[-2] \circ \varphi^{-1}[-2])[2] \circ \varphi \circ -) = \\ &= a^*(\varphi \circ \lambda \circ -) = (m(a))\lambda, \end{aligned}$$

which proves the claim. □

6 The Boundary Algebra *B*

In this section we discuss an algebra $B = B(k, n)$ which was introduced in [12] in order to categorify the cluster algebra structure of the coordinate ring of the Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$. This algebra also plays a prominent role in [4].

Let us consider a $\mathbb{Z}/n\mathbb{Z}$-grading on $\mathbb{C}[x, y]$ by $\deg x = 1$ and $\deg y = -1$. Thus the element $x^k - y^{n-k}$ is homogeneous of degree k, and we can consider graded modules over $R = \mathbb{C}[x, y]/(x^k - y^{n-k})$. Denote degree shift on $\mathrm{mod}^{\mathbb{Z}/n\mathbb{Z}} R$ by (1), and define B to be the algebra

$$B = \mathrm{End}_R^{\mathbb{Z}/n\mathbb{Z}}\left(\bigoplus_{i\in\mathbb{Z}/n\mathbb{Z}} R(i)\right).$$

We can realise B as a quiver algebra as follows. Consider the quiver with vertex set $[n]$, and arrows $x_i : i - 1 \to i$ and $y_i : i \to i - 1$ for each $i \in [n]$. Call $x = \sum_i x_i$ and $y = \sum_i y_i$. Then B is isomorphic to the quotient of the path algebra over $\mathbb{C}$ of this quiver by the ideal generated by the relations $xy = yx$ and $x^k = y^{n-k}$. Thus B is a quotient of the preprojective algebra of type $\tilde{A}_{n-1}$ by the relation $x^k = y^{n-k}$.

We also need to introduce the completed algebra $\hat{B}$. This is the completion of B with respect to the ideal (x, y) [12, Remarks 3.1, 3.2 and 3.4]. Similarly, we write $\hat{R} = \mathbb{C}[[x, y]]/\overline{(x^k - y^{n-k})}$. The completion will turn out not to play an important role for us, due to Proposition 3.3.

The categories $\mathrm{mod}\, B$ and $\mathrm{mod}^{\mathbb{Z}/n\mathbb{Z}} R$ are equivalent, and similarly for $\hat{B}$ and $\hat{R}$. We can consider the category $\mathrm{CM}(B)$ of Cohen-Macaulay modules over B and the category $\mathrm{CM}^{\mathbb{Z}/n\mathbb{Z}}(R)$ of graded Cohen-Macaulay modules over R. These also turn out to be equivalent (cf. [12, Corollary 3.7]), and again the same holds for the completed algebras. The category $\mathrm{CM}(\hat{B})$ was studied in [12], where the authors show that the Frobenius category $\mathrm{Sub}\, Q_k$ used in [7] is a quotient of $\mathrm{CM}(\hat{B})$ by one indecomposable projective object. In this way, many facts about $\mathrm{CM}(\hat{B})$ and its stable category $\underline{\mathrm{CM}}(\hat{B})$ can be deduced from what is known about $\mathrm{Sub}\, Q_k$. In particular, we have

Proposition 6.1 *The category $\underline{\mathrm{CM}}(\hat{B})$ is 2-Calabi-Yau.*

Proof This follows from [12, Corollary 4.6] and [7, Proposition 3.4]. □

Now we describe some additional properties of B, which are insensitive to the completion. We refer to [12] for a detailed discussion of the relationship between B and $\hat{B}$.

There is an automorphism $\Phi : B \to B$ given by mapping $e_i \mapsto e_{i+k}$, $x_i \mapsto x_{i+k}$ and $y_i \mapsto y_{i+k}$. The same function is also an automorphism Φ of B^{opp}.

The center of B is $Z = \mathbb{C}[t] \subseteq B$, where $t = xy$. The algebra B is finitely generated over Z, and the category $\mathrm{CM}(B)$ consists exactly of the finitely generated B-modules that are free over Z. Such a module corresponds to a representation of the quiver of B with at every vertex a free Z-module of the same rank [12, §3]. Following [12, Definition 3.5], we say that a B-module has rank d if it has rank nd as a Z-module. Rank is additive over short exact sequences (cf. [12, §3]), so in particular rank 1 modules are indecomposable.

Definition 6.2 [12, Definition 5.1] For each $I \in \binom{[n]}{k}$, define the B-module L_I of rank 1 by the following representation of the quiver: at every vertex i we have a copy U_i of Z, and

$$x_i : U_{i-1} \to U_i \text{ acts as multiplication by 1 if } i \in I, \text{ and by } t \text{ else,}$$
$$y_i : U_i \to U_{i-1} \text{ acts as multiplication by } t \text{ if } i \in I, \text{ and by 1 else.}$$

Similarly, the center of $\hat{B}$ is $Z = \mathbb{C}[[t]]$, and all the above holds for $\hat{B}$. In particular, we will use the notation Z for the center and L_I for the modules defined above, both for B and $\hat{B}$.

Such modules can be represented by lattice diagrams as in Fig. 4, where the black dots on column i represent the monomials $1, t, t^2, \ldots$ in the corresponding U_i, the action of x_i (y_i) is denoted by a rightward (leftward) arrow labelled i, and the edges of the figure are identified along the dotted lines. The label I can be read off from the arrows pointing to the right on the top profile of the diagram.

Every rank 1 module in $\mathrm{CM}(B)$ (or $\mathrm{CM}(\hat{B})$) is of this form for some (unique) set $I \in \binom{[n]}{k}$ [12, Proposition 5.2].

Definition 6.3 From the definition of the modules L_I, it is clear that the effect of twisting by Φ is the same as relabelling the columns of the lattice diagram. In other words, we have that $L_I \cong {}_\Phi L_{I+k}$ in a canonical way. We will denote by $\varphi_I : L_I \to {}_\Phi L_{I+k}$ this canonical isomorphism (given by identifying lattice diagrams).

Remark 6.4 There are some important differences between $\mathrm{CM}(B)$ and $\mathrm{CM}(\hat{B})$. In both categories, we have for every $i \in [n]$ that $\mathcal{P}_i \cong \mathcal{I}_{i+k} \cong L_{i+1,\ldots,i+k}$, cf. [4, Remark 7.2]. These are the only indecomposable projective objects in $\mathrm{CM}(\hat{B})$. Observe that we do not know whether this holds in $\mathrm{CM}(B)$, cf. [12, paragraph after Remark 3.4].

We need some notation about morphisms between the modules L_I (cf. [4, Lemma 7.4]). The spaces $\mathrm{Hom}_B(L_I, L_J)$ (respectively $\mathrm{Hom}_{\hat{B}}(L_I, L_J)$) are free modules of rank 1 over $\mathbb{C}[t]$ (respectively $\mathbb{C}[[t]]$), generated by the morphism $g_{JI} : L_I \to L_J$ corresponding to an embedding of lattice diagrams such that $\dim_{\mathbb{C}} \operatorname{coker}(g_{JI})$ is minimal. The map $g_{JI}t^N$ corresponds to the embedding where the diagram of L_I is shifted downwards N steps.

There is a functor $\mathcal{F}$ on $\mathrm{CM}(B)$ given by $M \mapsto {}_\Phi M$ on objects and $f \mapsto f$ on morphisms. We have that $\mathcal{F}(L_I) \cong L_{I-k}$ via the canonical isomorphism φ_{I-k}. It is clear from the definition of the morphisms g_{JI} that $\mathcal{F}(g_{JI}) = g_{JI} = \varphi_J^{-1} \circ g_{J-k,I-k} \circ \varphi_I$. Notice that $\mathcal{F}$ is the identity on morphisms, but the map g_{JI} changes name since we have relabelled the

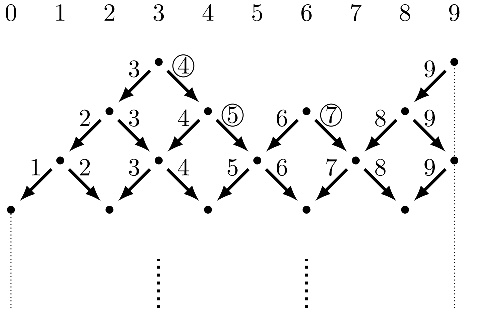

Fig. 4 The module L_{457} in the case $k = 3, n = 9$

basis elements of both domain and codomain. We will sometimes treat the canonical isomorphisms as identifications and write $\mathcal{F}(g_{JI}) = g_{J-k,I-k}$. Observe that via the equivalence $\mathrm{CM}(B) \to \mathrm{CM}^{\mathbb{Z}/n\mathbb{Z}}(R)$, the functor $\mathcal{F}$ is mapped to degree shift by $-k$ (cf. [2, Proposition 3.15]). Again, we define a functor on $\mathrm{CM}(\hat{B})$ in the same way, and call it $\mathcal{F}$ as well.

Definition 6.5 We denote by $\mathcal{B}$ the full additive subcategory of $\underline{\mathrm{CM}}(\hat{B})$ generated by $\left\{L_I \mid I \in \binom{[n]}{k}\right\}$.

Even though $\underline{\mathrm{CM}}(\hat{B})$ is triangulated, we remark that $\mathcal{B}$ is not a triangulated subcategory of $\underline{\mathrm{CM}}(\hat{B})$. However, we can explicitly describe the Serre functor [2] of $\underline{\mathrm{CM}}(\hat{B})$ (see also [2, Proposition 3.15]). It turns out that inside $\underline{\mathrm{CM}}^{\mathbb{Z}/n\mathbb{Z}}(R)$ there is an isomorphism of functors $[2] \cong (-k)$, which in our setting translates into the following:

Theorem 6.6 [6, Theorem 3.22] *There is an isomorphism of functors $\mathcal{F} \cong [2]$ on $\underline{\mathrm{CM}}(\hat{B})$.*

Remark 6.7 It follows in particular that $\mathcal{B}$ is closed under the Serre functor [2].

Remark 6.8 The statement that $L_I \cong L_{I+k}[2]$ appears also in [1, Proposition 2.7], and is implicitly used in [12, §7] in some specific cases. For our purposes, we will only need that $[2] \cong \mathcal{F}$ as functors on $\mathcal{B}$. We provide a direct proof of the latter fact.

Proof that $\mathcal{F} \cong [2]$ on $\mathcal{B}$ Let us denote by $[k]$ the interval $\{1, 2, \ldots, k\} \subseteq [n]$, and for $x \in [n]$ let us denote by $x + [k]$ the cyclic interval $\{x + 1, \ldots, x + k\} \subseteq [n]$.

Let us first prove that for every $I \in \binom{[n]}{k}$, there is an exact sequence in $\mathrm{CM}(\hat{B})$

$$0 \longrightarrow L_{I+k} \xrightarrow{\ f\ } \bigoplus_{v\in V} L_{v+[k]} \xrightarrow{\ \partial\ } \bigoplus_{u\in U} L_{u+[k]} \xrightarrow{\ h\ } L_I \longrightarrow 0$$

where $U = \{u \notin I \mid u + 1 \in I\}$ and $V = \{v \in I \mid v + 1 \notin I\}$. The map f is given by

$$f = \left(g_{v+[k],I+k}\right)_{v\in V},$$

the map h is given by

$$h = \left(g_{I,u+[k]}\right)_{u\in U},$$

and the map ∂ is given by

$$\partial = \left(\partial_{uv}\right)_{u\in U, v\in V}$$

with

$$\partial_{uv} = \begin{cases} g_{u+[k],v+[k]} & \text{if } u \text{ is the predecessor of } v \text{ in the cyclic order on } U \cup V; \\ -g_{u+[k],v+[k]} & \text{if } u \text{ is the successor of } v \text{ in the cyclic order on } U \cup V; \\ 0 & \text{otherwise.} \end{cases}$$

In particular we mean that $\partial = 0$ if L_I is projective. To prove the assertion that the above sequence is exact, observe that

$$\bigoplus_{v\in V} L_{v+[k]} \xrightarrow{\ \partial\ } \bigoplus_{u\in U} L_{u+[k]} h \xrightarrow{\ -\ } L_I \longrightarrow 0$$

is a projective presentation of L_I (cf. [12, Proposition 5.6]), and similarly

$$0 \longrightarrow L_{I+k} \xrightarrow{\ f\ } \bigoplus_{v\in V} L_{v+[k]} \xrightarrow{\ \partial\ } \bigoplus_{u\in U} L_{u+[k]}$$

is an injective presentation of L_{I+k}.

Now let us consider two k-element subsets I, I', and the corresponding exact sequences. We will construct a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & L_{I+k} \arrow[r, "f"] \arrow[d, "g_{I'+k,I'+k}"] & \bigoplus_{v\in V} L_{v+[k]} \arrow[r, "\partial"] \arrow[d, "N"] & \bigoplus_{u\in U} L_{u+[k]} \arrow[r, "h"] \arrow[d, "M"] & L_I \arrow[r] \arrow[d, "g_{I'I}"] & 0 \\
0 \arrow[r] & L_{I'+k} \arrow[r, "f'"] & \bigoplus_{v'\in V'} L_{v'+[k]} \arrow[r, "\partial'"] & \bigoplus_{u'\in U'} L_{u'+[k]} \arrow[r, "h'"] & L_{I'} \arrow[r] & 0.
\end{tikzcd}$$

By the definition of the triangulated structure on $\underline{\mathrm{CM}}(\hat{B})$, this means that $g_{I'+k,I+k}[2] = g_{I'I} \cong \mathcal{F}(g_{I'+k,I+k})$. Since all morphism spaces in $\mathcal{B}$ are generated by maps of this form (in particular, recall that $g_{II} = \mathrm{id}_{L_I}$), this will be enough to prove the assertion that $[2] \cong \mathcal{F}$ on $\mathcal{B}$.

Let us fix some more notation to simplify the construction. We will drop the $+[k]$ in indices to avoid clogging the formulas. For instance, we will write L_u for $L_{u+[k]}$ and similarly g_{uI} for $g_{u+[k],I}$. For the cyclic orders on $U \cup V$ and on $U' \cup V'$, we write $\mathfrak{s}$ and $\mathfrak{p}$ for the successor and predecessor functions.

Let us fix $u \in U$. We write $(L_{I'})_u$ for the Z-module of rank 1 corresponding to vertex u inside $L_{I'}$. The generator of $(L_{I'})_u$ as a Z-module is in the image via h' of either one or two of the $L_{u'}$. If there is only one such $L_{u'}$, then define $\mathfrak{u}'(u) = u'$. If there are two such $L_{u'_1}$ and $L_{u'_2}$ (this happens if and only if $|U'| \geq 2$ and $u \in V'$), with $u'_1 < u < u'_2$, then define $\mathfrak{u}'(u) = u'_1$. In other words, $\mathfrak{u}'(u)$ is the unique element of U' such that $\mathfrak{p}(\mathfrak{u}'(u)) < u \leq \mathfrak{s}(\mathfrak{u}'(u))$. By construction we have $g_{I'\mathfrak{u}'(u)} \circ g_{\mathfrak{u}'(u)u} = g_{I'u}$. We define $d(u)$ by the equation $g_{I'u}t^{d(u)} = g_{I'I} \circ g_{Iu}$.

Dually, let us fix $v' \in V'$. The generator of $(L_{I+k})_{v'+k}$ as a Z-module is mapped via f to a Z-module generator in either one or two of the L_v. If there is only one such L_v, then define $\mathfrak{v}(v') = v$. If there are two such L_{v_1} and L_{v_2} (this happens if and only if $|V| \geq 2$ and $v' \in U$), with $v_1 < v' < v_2$, then define $\mathfrak{v}(v') = v_2$. In other words, $\mathfrak{v}(v')$ is the unique element of V such that $\mathfrak{p}(\mathfrak{v}(v')) \leq v' < \mathfrak{s}(\mathfrak{v}(v'))$. By construction we have $g_{v'\mathfrak{v}(v')} \circ g_{\mathfrak{v}(v'),I+k} = g_{v',I+k}$. We define $d(v')$ by the equation $g_{v',I+k}t^{d(v')} = g_{v',I'+k} \circ g_{I'+k,I+k}$.

Now we can define maps $M : \bigoplus_{u\in U} L_u \to \bigoplus_{u'\in U'} L_{u'}$ and $N : \bigoplus_{v\in V} L_v \to \bigoplus_{v'\in V'} L_{v'}$ by setting

$$M_{u'u} = \begin{cases} g_{u'u}t^{d(u)}, & \text{if } u' = \mathfrak{u}'(u); \\ 0, & \text{otherwise} \end{cases}$$

and

$$N_{v'v} = \begin{cases} g_{v'v}t^{d(v')}, & \text{if } v = \mathfrak{v}(v'); \\ 0, & \text{otherwise.} \end{cases}$$

Let us check that the right square commutes, the left square being similar. We have

$$\begin{aligned}(h' \circ M)_u &= \sum_{u'\in U'} g_{I'u'} \circ M_{u'u} = g_{I'\mathfrak{u}'(u)} \circ g_{\mathfrak{u}'(u)u}t^{d(u)} = \\ &= g_{I'u}t^{d(u)} = g_{I'I} \circ g_{Iu} = \\ &= (g_{I'I} \circ h)_u.\end{aligned}$$

Let us now consider the middle square. We have

$$(M \circ \partial)_{u'v} = M_{u'\mathfrak{p}(v)} \circ g_{\mathfrak{p}(v)v} - M_{u'\mathfrak{s}(v)} \circ g_{\mathfrak{s}(v)v}$$

and

$$(\partial' \circ N)_{u'v} = g_{u'\mathfrak{s}(u')} \circ N_{\mathfrak{s}(u')v} - g_{u'\mathfrak{p}(u')} \circ N_{\mathfrak{p}(u')v}.$$

There are four cases to consider.

Case 1. Let us first assume that $\mathfrak{u}'(\mathfrak{p}(v)) = \mathfrak{u}'(\mathfrak{s}(v)) = u'$. In this case we have

$$(M \circ \partial)_{u'v} = g_{u'\mathfrak{p}(v)} \circ g_{\mathfrak{p}(v)v} t^{d(\mathfrak{p}(v))} - g_{u'\mathfrak{s}(v)} \circ g_{\mathfrak{s}(v)v} t^{d(\mathfrak{s}(v))}.$$

In particular, $(M \circ \partial)_{u'v} = (t^a - t^b) g_{u'v}$ for some $a, b \geq 0$. From $h' \circ M \circ \partial = 0$ we get then that $a = b$ and thus that $(M \circ \partial)_{u'v} = 0$. Now since $\mathfrak{p}(u') < \mathfrak{p}(v) \leq \mathfrak{s}(u')$ and $\mathfrak{p}(u') < \mathfrak{s}(v) \leq \mathfrak{s}(u')$, we must have that either $\mathfrak{v}(\mathfrak{p}(u')) = \mathfrak{v}(\mathfrak{s}(u')) = v$ or $\mathfrak{v}(\mathfrak{p}(u')) \neq v \neq \mathfrak{v}(\mathfrak{s}(u'))$. In the first case,

$$(\partial' \circ N)_{u'v} = g_{u'\mathfrak{s}(u')} \circ g_{\mathfrak{s}(u')v} t^{d(\mathfrak{s}(u'))} - g_{u'\mathfrak{p}(u')} \circ g_{\mathfrak{p}(u')v} t^{d(\mathfrak{p}(u'))}$$

and as above we can argue that this is 0. In the second case, $(\partial' \circ N)_{u'v} = 0 - 0$ directly.

Case 2. In a similar way we can argue that $(M \circ \partial)_{u'v} = (\partial' \circ N)_{u'v} = 0$ whenever $\mathfrak{u}'(\mathfrak{p}(v)) \neq u' \neq \mathfrak{u}'(\mathfrak{s}(v))$.

Case 3. Let us now assume that $\mathfrak{u}'(\mathfrak{p}(v)) = u' \neq \mathfrak{u}'(\mathfrak{s}(v))$. This means that $\mathfrak{p}(u') < \mathfrak{p}(v) \leq \mathfrak{s}(u') < \mathfrak{s}(v)$, so we obtain that $\mathfrak{v}(\mathfrak{s}(u')) = v \neq \mathfrak{v}(\mathfrak{p}(u'))$. Thus

$$(M \circ \partial)_{u'v} = g_{u'\mathfrak{p}(v)} \circ g_{\mathfrak{p}(v)v} t^{d(\mathfrak{p}(v))}$$

and

$$(\partial' \circ N)_{u'v} = g_{u'\mathfrak{s}(u')} \circ g_{\mathfrak{s}(u')v} t^{d(\mathfrak{s}(u'))}.$$

We need to make some observations (cf. [1, Proposition 2.7] for a pictorial interpretation). First, the maps $g_{Iv} \circ g_{v,I+k}$ are all equal, and we can call them ι_I. The maps $g_{Iu} \circ g_{u,I+k}$ are also all equal to ι_I. Defining $\iota_{I'}$ in the same way, we remark that $g_{I'I} \circ \iota_I = \iota_{I'} \circ g_{I'+k,I+k}$.

With these observations, we can write

$$\begin{aligned}
g_{I'u'} \circ (M \circ \partial)_{u'v} \circ g_{v,I+k} &= g_{I'u'} \circ g_{u'\mathfrak{p}(v)} \circ g_{\mathfrak{p}(v)v} t^{d(\mathfrak{p}(v))} \circ g_{v,I+k} = \\
&= g_{I'I} \circ g_{Iv} \circ g_{v,I+k} = \\
&= g_{I'I} \circ \iota_I = \\
&= \iota_{I'} \circ g_{I'+k,I+k} = \\
&= g_{I'u'} \circ g_{u',I'+k} \circ g_{I'+k,I+k} = \\
&= g_{I'u'} \circ g_{u'\mathfrak{s}(u')} \circ g_{\mathfrak{s}(u')v} t^{d(\mathfrak{s}(u'))} \circ g_{v,I+k} = \\
&= g_{I'u'} \circ (\partial' \circ N)_{u'v} \circ g_{v,I+k}.
\end{aligned}$$

Since both $(M \circ \partial)_{u'v}$ and $(\partial' \circ N)_{u'v}$ are equal to a power of t times $g_{u'v}$, we conclude that they must be equal.

Case 4. The case $\mathfrak{u}'(\mathfrak{p}(v)) \neq u' = \mathfrak{u}'(\mathfrak{s}(v))$ is similar to Case 3. We conclude that the middle square and thus the whole diagram commutes, and so we are done. □

7 Cluster Tilting in $\underline{\mathrm{CM}}(\hat{B})$

There is a strong relationship between combinatorics of Postnikov diagrams and homological algebra in $\underline{\mathrm{CM}}(\hat{B})$. We are interested in cluster tilting objects in the Frobenius category $\mathrm{CM}(\hat{B})$ and in the 2-Calabi-Yau category $\underline{\mathrm{CM}}(\hat{B})$, and these are the same objects. To be more precise, there is a bijection between isomorphism classes of basic cluster tilting objects in $\mathrm{CM}(\hat{B})$ and in $\underline{\mathrm{CM}}(\hat{B})$, given by adding or removing the projective indecomposables.

The noncrossing property introduced in Section 4 corresponds to Ext-vanishing in $\mathrm{CM}(\hat{B})$.

Proposition 7.1 [12, Proposition 5.6] *Let $I, J \in \binom{[n]}{k}$. Then $\mathrm{Ext}^1_B(L_I, L_J) = 0$ if and only if I and J are noncrossing.*

Let D be a reduced (k, n)-Postnikov diagram, and let $\mathbb{I} = \mathbb{I}(D)$. Define the $\hat{B}$-module T by

$$T = \bigoplus_{I \in \mathbb{I}} L_I.$$

We also denote by T the B-module defined in the same way. This abuse of notation is justified by the fact that the stable endomorphism algebra of T is the same regardless of the completion, as we will see.

The following theorem was proved in [12].

Theorem 7.2 *For any reduced (k, n)-Postnikov diagram, the module T defined above is a cluster tilting object in* $\mathrm{CM}(\hat{B})$.

Proof Since $\mathbb{I}$ is a maximal noncrossing collection, it follows by Proposition 7.1that T is a maximal rigid object in $\mathrm{CM}(\hat{B})$. Since $\mathrm{CM}(\hat{B})$ is known to have at least one cluster tilting object, this is equivalent to T being cluster tilting (cf. [12, Remark 4.8]). □

We remark that T is also a maximal rigid object in $\mathrm{CM}(B)$, but we do not know whether it is actually cluster tilting.

Remark 7.3 Any maximal noncrossing collection contains the n cyclic intervals of length k. Remark 4.4 says that the labels of the n boundary regions of a Postnikov diagram are precisely these n cyclic intervals. Indeed, any cluster tilting object in $\mathrm{CM}(\hat{B})$ has as summands the n indecomposable projective-injective objects, which are labelled by such intervals.

Theorem 7.4 [4, Theorem 10.3 and Theorem 11.2] *Let D be a reduced (k, n)-Postnikov diagram, let T be as above and let $A(D) = \wp(Q, W, F)$ be as in Section 4. Then there exists a unique isomorphism $A(D) \to \mathrm{End}_B(T)$ such that the vertex I of Q is mapped to id_{L_I} and any arrow $I \to J$ in Q is mapped to the morphism $g_{JI} : L_I \to L_J$. Moreover, this induces an isomorphism $\hat{A}(D) = \hat{\wp}(Q, W, F) \to \mathrm{End}_{\hat{B}}(T)$.*

We call $g : A \to \mathrm{End}_B(T)$ this isomorphism. In particular, the frozen vertices of Q correspond to the indecomposable projective B-modules, and B^{opp} is identified with $eAe \subseteq A$ [4, Corollary 10.4]. In this article, we focus on the study of the algebra $\Lambda = A/AeA$. This corresponds to quotienting out endomorphisms of T factoring through its projective summands, thus moving to the stable category.

Lemma 7.5 *The isomorphism $g : A \to \mathrm{End}_B(T)$ induces an isomorphism $\underline{g} : A/AeA \to \underline{\mathrm{End}}_B(T)$. In the same way, the isomorphism $\hat{A} \cong \mathrm{End}_{\hat{B}}(T)$ induces an isomorphism $\hat{A}/\hat{A}e\hat{A} \cong \underline{\mathrm{End}}_{\hat{B}}(T)$.*

Proof We give the proof for the non-complete case; the other case is similar. We have $T \cong T' \oplus P$, where P is the sum of the indecomposable projective B-modules $\mathcal{P}_i$. Call E the subset of $\mathrm{End}_B(T)$ consisting of maps that factor through P. There is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & AeA \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "g"] & A/AeA \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & E \arrow[r] & \mathrm{End}_B(T) \arrow[r] & \underline{\mathrm{End}}_B(T) \arrow[r] & 0 \end{tikzcd}$$

where the two leftmost vertical maps are isomorphisms, thus the claim is proved. □

The following results will justify our claims that the completion does not play a big role in our setting.

Proposition 7.6 *We have* $A/AeA \cong \wp(\underline{Q}, \underline{W}) \cong \hat{A}/\hat{A}e\hat{A}$.

Proof Let us first prove the first isomorphism. By Corollary 3.4, it is enough to prove that every sufficiently long path in Q is equivalent in A to a path through a frozen vertex. By [4, Corollary 9.4], we have that a basis of $e_J A e_I$ is given by the set

$$\left\{u^N p_{IJ} \mid N \in \mathbb{N}\right\}$$

where u is as in Definition 4.6 and p_{IJ} is a chosen path from I to J that is not equivalent to a path containing a cycle. There is a unique equivalence class of paths from I to J containing such an element. This basis is mapped via g to the basis

$$\left\{t^N g_{JI} \mid N \in \mathbb{N}\right\}$$

of $\mathrm{Hom}_B(L_I, L_J)$.

Now observe that paths of a fixed degree in u have bounded length, so for any d we can find a path with degree larger than d. Translated into maps from L_I to L_J, it is then enough to prove that every map of the form $g_{JI}t^N$ for $N \gg 0$ factors through a map g_{PI} for a projective L_P. Given the description of maps of the form $g_{JI}t^N$ as embeddings of lattice diagrams, this is clear.

To conclude, observe that the above argument implies that the two-sided ideal $\hat{A}e\hat{A}$ is closed in $\hat{A}$, so the second isomorphism follows. □

From Theorem 7.4, Lemma 7.5 and Proposition 7.6, we get:

Corollary 7.7 *There is an algebra isomorphism*

$$\underline{\mathrm{End}}_B(T) \cong \underline{\mathrm{End}}_{\hat{B}}(T).$$

In view of Corollary 7.7, we can essentially ignore the completions. In particular, all statements about T that depend on the triangulated or Calabi-Yau structure of $\underline{\mathrm{CM}}(B)$ (such as mutation and suspensions) can be carried out in $\underline{\mathrm{CM}}(\hat{B})$ instead, without affecting $\underline{\mathrm{End}}_B(T)$ and therefore Λ.

The following results about $\mathrm{CM}(\hat{B})$ coming from the combinatorics of Postnikov diagrams still hold true for $\mathrm{CM}(B)$, if we replace “cluster tilting” by “maximal rigid”.

Lemma 7.8 *For any fixed (k, n) there is a bijection*

$$\left\{\begin{matrix}\textit{Basic cluster tilting objects in } \mathrm{CM}(\hat{B}) \\ \textit{that lie in } \mathcal{B}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{Maximal noncrossing} \\ \textit{collections of} \\ \textit{elements of } \binom{[n]}{k}\end{matrix}\right\}.$$

Proof Since the modules L_I are indecomposable, they are precisely the indecomposable objects in $\mathcal{B}$. It follows that maximal rigid objects in $\mathrm{CM}(\hat{B})$ that lie in $\mathcal{B}$ correspond precisely to maximal noncrossing collections. $\square$

Combining this with Theorem 4.3 we get

Proposition 7.9 *Basic cluster tilting objects in* $\mathrm{CM}(\hat{B})$ *(respectively in* $\underline{\mathrm{CM}}(\hat{B})$*) that lie in* $\mathcal{B}$ *are precisely those contructed as above from reduced* (k, n)*-Postnikov diagrams.*

There are various notions of mutation for the various objects we are considering, and in a sense they all correspond to each other. The rest of this section is devoted to making this statement a bit more precise.

There is a well-defined mutation of cluster tilting objects in $\mathrm{CM}(\hat{B})$ [12, Remark 4.8]. Namely, if $X \oplus T$ is a cluster tilting object and X is indecomposable nonprojective then there is a unique indecomposable nonprojective $Y \ncong X$ such that $Y \oplus T$ is cluster tilting. Now if T is cluster tilting in $\mathrm{CM}(\hat{B})$ and moreover $T \in \mathcal{B}$, then $T = \bigoplus_{I \in \mathbb{I}} L_I$ for some maximal noncrossing collection $\mathbb{I}$. Suppose that $I \in \mathbb{I}$ is not a cyclic interval of $[n]$ (i.e. not the label of a projective $\hat{B}$-module). Then, under some condition, there is a unique I' by which we can replace I so that $\mathbb{I}\setminus\{I\}\cup\{I'\}$ is a maximal noncrossing collection. The precise description of I' is rather cumbersome, and can be found for instance in [14, Theorem 1.4]. If we start from the cluster tilting object $T = \bigoplus_{J \in \mathbb{I}} L_J \in \mathcal{B}$, and we mutate it at L_I, the new cluster tilting object will be $\bigoplus_{J \in \mathbb{I}'} L_J$ by Proposition 7.9.

There is a combinatorial interpretation of mutation of cluster tilting objects (or more directly of maximal noncrossing collections) in terms of Postnikov diagrams. This is given by the notion of *geometric exchange* on a Postnikov diagram, i.e. applying the local operation depicted in Fig. 5, followed by untwisting and boundary untwisting moves to make the new Postnikov diagram reduced.

Notice that the labels of vertices do not change except at the chosen vertex. The label I' is precisely the only k-element set which is not I which makes the collection of labels noncrossing. The effect on the corresponding quiver is almost Fomin-Zelevinsky mutation.

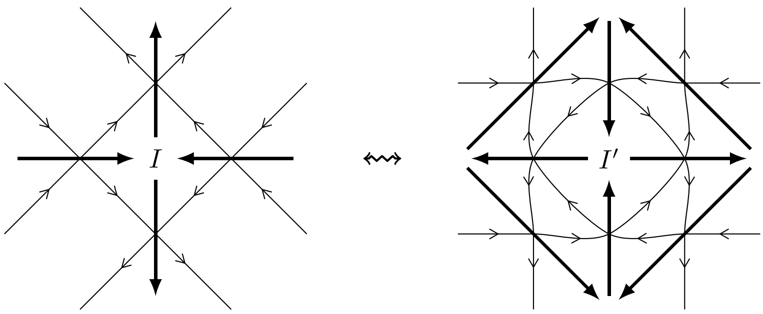

Fig. 5 Geometric exchange and the corresponding operation on the quiver

The step of removing any new 2-cycles must be replaced as follows: remove any new 2-cycle consisting of non-boundary arrows, then for every 2-cycle consisting of a boundary arrow and a non-boundary arrow, remove the boundary arrow and treat the non-boundary arrow as a new boundary arrow. This corresponds to the effect of applying a boundary untwisting move, as opposed to a "normal" untwisting move (cf. [4, Lemma 12.1]). This mutation rule also coincides with mutation of ice quivers with potential presented in [16]. If we restrict our attention to the quiver $\underline{Q}$, this difference disappears (since arrows between frozen vertices are not arrows in $\underline{Q}$).

By the above discussion, the notions of mutation of Postnikov diagrams (i.e. geometric exchange), of cluster tilting objects in $\underline{\mathrm{CM}}(\hat{B})$, and of quivers with potential all correspond to each other when they make sense. We remark that sometimes mutation of a cluter tilting object in $\mathcal{B}$ will produce a cluster tilting object which does not lie in $\mathcal{B}$, and that will happen precisely when geometric exchange is not possible (because the chosen vertex does not have valency 4). The correspondence between mutation of cluster tilting objects and quivers with potential is a widespread phenomenon, see for instance [3].

In particular, we can read off mutation of cluster tilting objects in $\mathrm{CM}(\hat{B})$ (respectively in $\underline{\mathrm{CM}}(\hat{B})$) from the Postnikov diagram D, from the quiver Q, or from the collection $\mathbb{I}$. In Figs. 6 and 7, we illustrate the geometric exchange at 134 of the Postnikov diagram of Fig. 2 and the corresponding mutation of the quiver with potential. Vertex 134 is mutated to 245. We can deduce that mutation is transitive on cluster tilting objects that lie in $\mathcal{B}$:

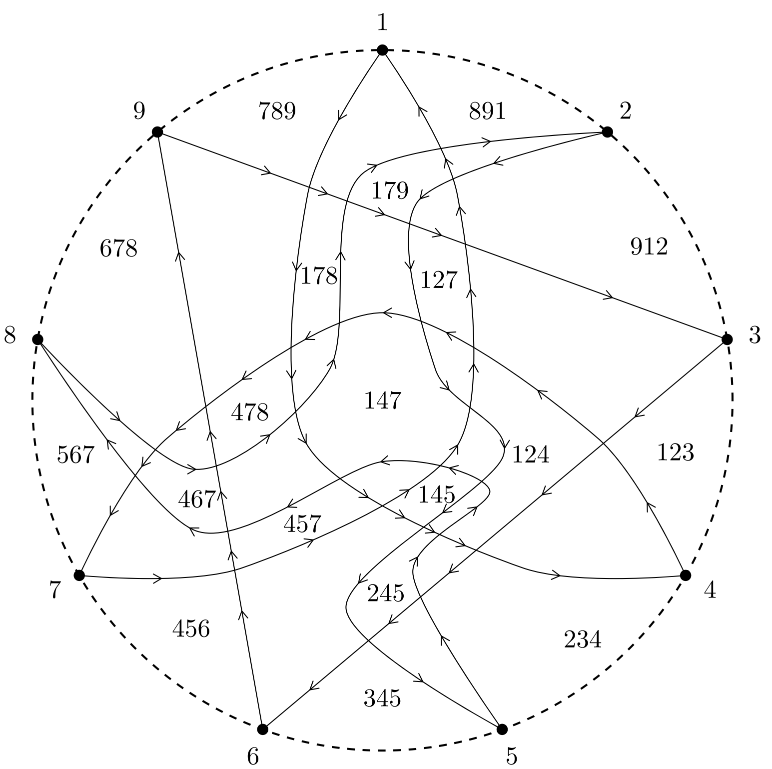

Fig. 6 The geometric exchange at 134 of the Postnikov diagram of Fig. 2

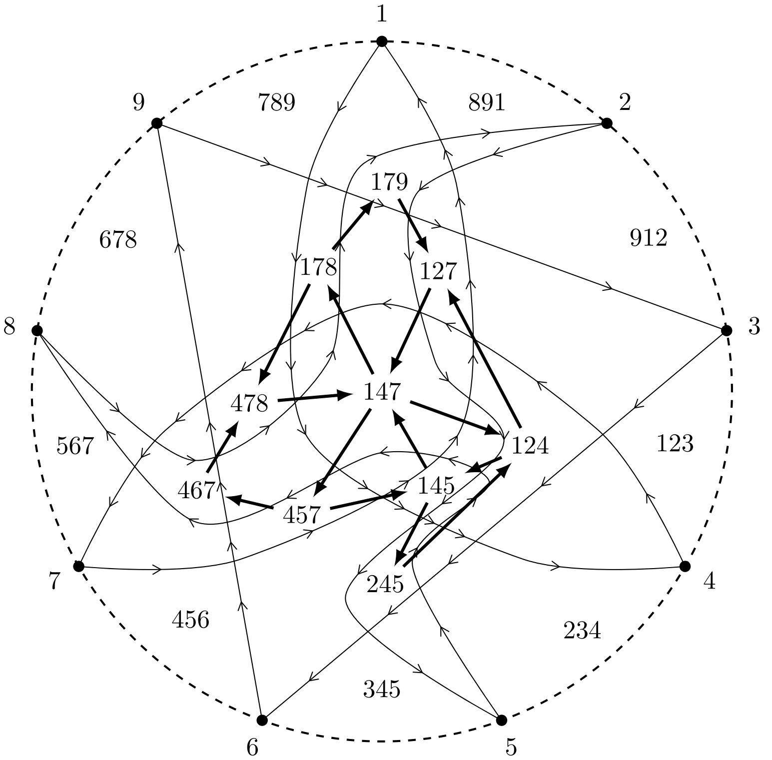

Fig. 7 The quiver $\mu_{134}(\underline{Q})$, where Q is the quiver of Fig. 3

Theorem 7.10 [15, Theorem 13.4] *Any two reduced* (k, n)*-Postnikov diagrams are related by a sequence of geometric exchange, twisting and untwisting moves.*

Corollary 7.11 *Any two basic cluster tilting objects in* $\mathrm{CM}(\hat{B})$ *(or* $\underline{\mathrm{CM}}(\hat{B})$*) that lie in* $\mathcal{B}$ *are related by a sequence of mutations.*

Remark 7.12 Given two cluster tilting objects as above T, T', one can go from T to T' via a sequence of quiver mutations *at vertices of valency 4*. Applying arbitrary mutations to the quiver can cause indecomposable summands of rank ≥ 2 to appear in the cluster tilting object.

In [9], the authors discuss the concept of *planar mutation*, which is a more restrictive notion than that of quiver mutation. It has the property of preserving planarity. Their definition allows mutation at internal vertices of valency 4, or at boundary vertices of valency at most 4. Mutation at an internal vertex of valency 4 of $\underline{Q}$ is precisely what is allowed by geometric exchange, but for boundary vertices the situation is different. Namely, we can mutate at the boundary vertices of $\underline{Q}$ if and only if they have valency 4 as vertices of Q, which is a stronger condition than having valency at most 4 in $\underline{Q}$.

8 Self-Injective Cluster Tilting Objects in $\underline{\mathrm{CM}}(B)$

We are now ready to state our main result. In this section, let D be a reduced (k,n)-Postnikov diagram.

Lemma 8.1 *The Postnikov diagram D is symmetric if and only if ${}_\Phi T \cong T$ as left B-modules.*

Proof Assume that D is symmetric. As left B-modules, we have

$$T = \bigoplus_{I\in\mathbb{I}} L_I \cong \bigoplus_{I\in\mathbb{I}} {}_\Phi L_{I+k} = \bigoplus_{I\in\mathbb{I}+k} {}_\Phi L_I = \bigoplus_{I\in\mathbb{I}} {}_\Phi L_I = {}_\Phi T$$

where we have used the isomorphism of Definition 6.3 and Lemma 4.5. On the other hand, there can be an isomorphism $\bigoplus_{I\in\mathbb{I}} {}_\Phi L_{I+k} \cong \bigoplus_{I\in\mathbb{I}+k} {}_\Phi L_{I+k}$ only if $\mathbb{I} = \mathbb{I}+k$, which by Lemma 4.5 implies that D is symmetric. □

In other words, $\mathcal{F}T \cong T$, and recall that $\mathcal{F} = [2]$ on $\mathcal{B} \subseteq \underline{\mathrm{CM}}(\hat{B})$. If we call $\varphi : T \to \mathcal{F}T$ the canonical isomorphism with components $\varphi_I : L_I \to {}_\Phi(L_{I+k})$ as in Definition 6.3, then there is an automorphism ψ of $\mathrm{End}_B(T)$ given by

$$\psi : a \mapsto \varphi \circ a \circ \varphi^{-1}.$$

By Remark 6.4, $\mathcal{F}$ sends projectives to projectives, so the automorphism ψ induces an automorphism of $\underline{\mathrm{End}}_B(T)$, which we still denote by ψ.

Theorem 8.2 *Let D be a reduced (k,n)-Postnikov diagram. Then D is symmetric if and only if the B-module $T \in \underline{\mathrm{CM}}(B)$ is self-injective. In this case the Nakayama permutation is given by $\sigma(I) = I - k$, and a Nakayama automorphism given by ψ.*

Proof By Corollary 7.7, T is self-injective as a B-module if and only if it is self-injective as a $\hat{B}$-module. Thus we can work in $\underline{\mathrm{CM}}(\hat{B})$, where we have a 2-Calabi-Yau structure. By Lemma 8.1 and Theorem 6.6, we have that D is symmetric if and only if $T \cong {}_\Phi T \cong T[2]$. Moreover, $T \cong T[2]$ if and only if T is self-injective, by Proposition 5.2. In this case, we have $L_I[2] \cong L_{I-k}$, which gives $\sigma : I \mapsto I - k$. Since $[2] = \mathcal{F}$ on the modules L_I, the map ψ we have defined is exactly the map in the statement of Proposition 5.3. Thus we conclude that ψ is a Nakayama automorphism of $\underline{\mathrm{End}}_{\hat{B}}(T) \cong \underline{\mathrm{End}}_B(T)$. □

Corollary 8.3 *Let D be a reduced (k,n)-Postnikov diagram. Then D is symmetric if and only if $(\underline{Q}, \underline{W})$ is a self-injective quiver with potential. In this case, the Nakayama permutation is $\sigma(I) = I - k$, induced by rotation by $2\pi k/n$, and a Nakayama automorphism of $\wp(\underline{Q}, \underline{W})$ is given by Ψ (see Section 4).*

Proof By Theorem 7.4 and Lemma 7.5 we have that $\underline{\mathrm{End}}_B(T) \cong \wp(\underline{Q}, \underline{W})$. The functor $\mathcal{F}$ on $\underline{\mathrm{CM}}(B)$ sends L_I to L_{I-k} and g_{JI} to $g_{J-k,I-k}$, so the automorphism ψ of Λ defined by twisting with the canonical isomorphism $\varphi : T \to \mathcal{F}T$ corresponds to the quiver automorphism sending vertex I to $I-k$ and an arrow $I \to J$ to an arrow $I-k \to J-k$. Thus the action of ψ on the quiver coincides with that of ρ, which in turn is the action on $\wp(\underline{Q}, \underline{W})$ of the automorphism Ψ. □

Remark 8.4 Strictly speaking, the rotation ρ acts on D only if D is chosen appropriately in the equivalence class modulo isotopy. In other words, the Nakayama automorphism acts by ρ on Q provided that Q is embedded in the plane with the embedding of Lemma 4.5.

Remark 8.5 The automorphism ψ of $\mathrm{End}_B(T)$ induces the automorphism Φ^{-1} on $B^{opp} \subseteq \mathrm{End}_B(T)$.

Definition 8.6 [9, Definition 4.1] Let $(\underline{Q}, \underline{W})$ be a self-injective quiver with potential constructed from a reduced (k, n)-Postnikov diagram. In this case, the Nakayama permutation acts on vertices by $\sigma : I \mapsto I - k$. Call $(I) = \{\sigma^j(I) \mid j \in \mathbb{Z}\}$ the orbit of I. Suppose that there are no arrows between any two vertices in (I). Then we define the *mutation at* (I) $\mu_{(I)}(\underline{Q}, \underline{W})$ to be the composition of mutations at the vertices in (I), applied to $\underline{Q}$. It is well defined since, by the assumption, it does not depend on the order of composition.

The following theorem is stated in greater generality in [9].

Theorem 8.7 [9, Theorem 4.2] *If $(\underline{Q}, \underline{W})$ is self-injective and I satisfies the above condition (allowing $\mu_{(I)}$ to be defined), then $\mu_{(I)}(\underline{Q}, \underline{W})$ is a self-injective quiver with potential with the same Nakayama permutation.*

In our setting, this result can be deduced immediately from Corollary 8.3 if I is mutable. Indeed, applying geometric exchange along a mutable orbit of ρ produces another symmetric (k, n)-Postnikov diagram, so the corresponding quiver is again self-injective with the same permutation.

Remark 8.8 By Theorem 7.10, any two symmetric reduced (k, n)-Postnikov diagrams are related by a sequence of geometric exchanges. However, we do not know whether they are related by a sequence of geometric exchanges along Nakayama orbits.

9 Cuts of Self-Injective Quivers with Potential

In this section we study the 2-representation finite algebras one can get from a self-injective quiver with potential. We want to use the results of [9], so again we need our Jacobian algebras to be completed.

Definition 9.1 For a quiver with potential (R, P), a *cut* is a set of arrows which contains exactly one arrow from every cycle in P. The quiver (R, P) *has enough cuts* if every arrow is contained in a cut.

We can define a grading on $\hat{\wp}(R, P)$ by giving degree 1 to the arrows in a cut C of R, since by definition the potential is then homogeneous of degree 1. The degree 0 part of $\hat{\wp}(R, P)$ is denoted $\hat{\wp}(R, P)_C$ and called the *truncated Jacobian algebra* of $\hat{\wp}(R, P)$ associated to C.

Recall that an algebra is called 2-representation finite if it has global dimension at most 2 and admits a cluster tilting module (cf. [11]). One reason to look at truncated Jacobian algebras is the following result (see for instance [9] for the definition of 3-preprojective algebras).

Theorem 9.2 [9, Theorem 3.11] *For any self-injective quiver with potential* (R, P) *and cut* C*, the truncated Jacobian algebra* $\hat{\wp}(R, P)_C$ *is 2-representation finite. All basic 2-representation finite algebras arise in this way. Moreover, the 3-preprojective algebra of* $\hat{\wp}(R, P)_C$ *is isomorphic to* $\hat{\wp}(R, P)$.

Now if D is a symmetric Postnikov diagram, by Theorem 8.2 the associated Jacobian algebra $\Lambda = \wp(\underline{Q}, \underline{W})$ is self-injective, and by Proposition 7.6 it is isomorphic to $\hat{\wp}(\underline{Q}, \underline{W})$. So for any cut C of $(\underline{Q}, \underline{W})$ the truncated Jacobian algebra Λ_C is 2-representation finite with 3-preprojective algebra isomorphic to Λ.

We need some notation for regions determined by Postnikov diagrams. A *boundary region* is a region whose boundary is alternating (ignoring the boundary of the disk) and has a piece of the boundary circle as part of its boundary. These are precisely the regions labeled by cyclic intervals. A *cyclic boundary region* is a cyclic region which shares an edge with a boundary region. On the level of Postnikov diagrams, a cut of $(\underline{Q}, \underline{W})$ is a set C of (non-boundary) crossings of strands such that

(1) for every crossing $c \in C$, the two cyclic regions adjacent to c are not both cyclic boundary regions, and
(2) every cyclic region which is not a cyclic boundary region is adjacent to exactly one crossing in C.

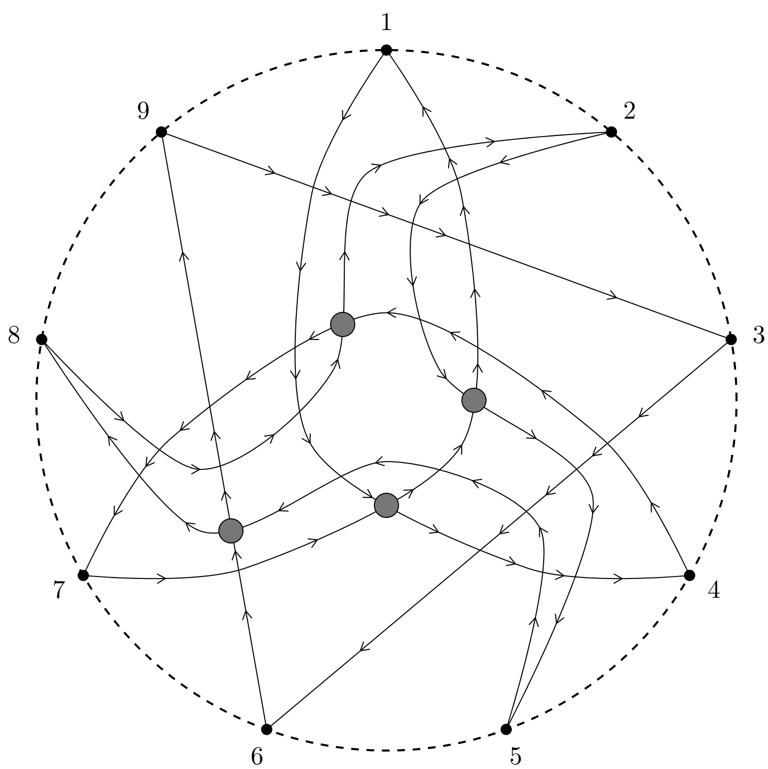

Fig. 8 A cut on a symmetric (3, 9)-Postnikov diagram

In Fig. 8 we illustrate such a cut, and in Fig. 9 we show the corresponding cut on the quiver $\underline{Q}$ (dotted arrows are arrows in the cut).

There is a notion of mutation of cuts that corresponds exactly to taking the quiver of the corresponding 2-APR tilt of Λ (see [9] for details).

Definition 9.3 [9, Definition 6.10] Let (R, P) be a quiver with potential with a cut C. A vertex x of R is a *strict source* if all arrows ending at x belong to C and all arrows starting at x do not belong to C. For a strict source x, call the *cut-mutation* $\mu_x^+(C)$ of R_1 the set of arrows we get by removing all arrows ending at x from C, and adding all arrows starting at x to C. Dually, we define a *strict sink* and the cut-mutation $\mu_x^-(C)$.

It is clear that cut-mutation transforms strict sources into strict sinks and vice-versa.

For a quiver with potential $(\underline{Q}, \underline{W})$ constructed from a Postnikov diagram, strict sources and strict sinks are precisely alternating regions such that every second crossing (except those with a boundary region) on their boundary is contained in the cut. Cut-mutation consists of replacing the crossings on the boundary of such alternating regions with their complement (again, ignoring the crossings with a boundary region). In Figs. 10 and 11 we illustrate $\mu_{457}^+(C)$ for the cut C of Figs. 8 and 9.

Quivers with potential obtained from Postnikov diagrams are by definition planar in the sense of [9, Definition 9.1]. We illustrate one application.

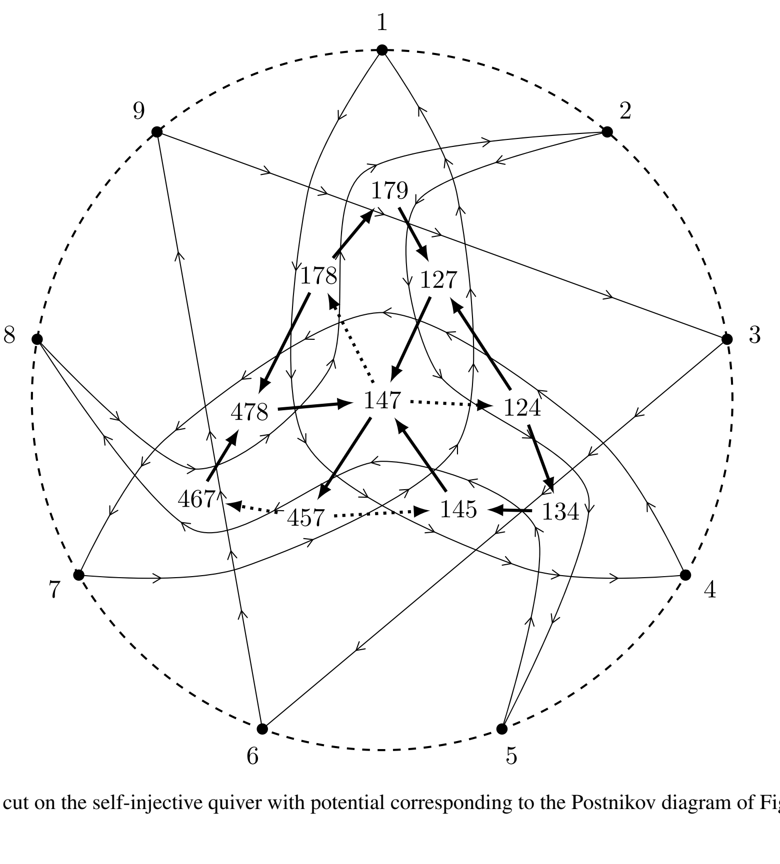

Fig. 9 A cut on the self-injective quiver with potential corresponding to the Postnikov diagram of Fig. 8

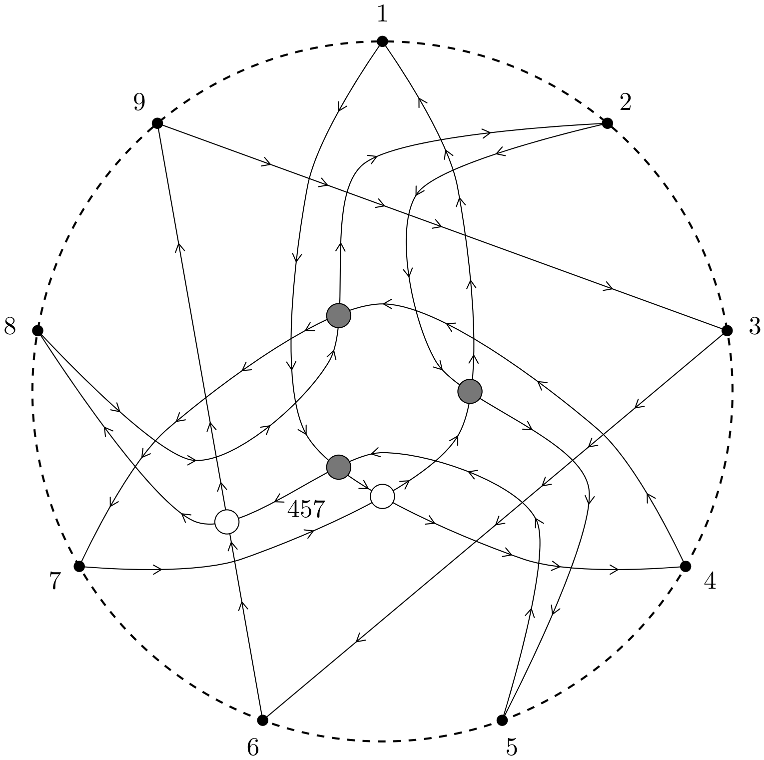

Fig. 10 The cut-mutation $\mu^{+}_{457}(C)$ of the cut in Fig. 8

Theorem 9.4 [9, Theorem 9.2] *Let* (R, P) *be a self-injective planar quiver with potential that has enough cuts. Then all truncated Jacobian algebras* $\wp(R, P)_C$ *are iterated 2-APR tilts of each other. In particular they are derived equivalent.*

The assumption is satisfied in our setting:

Proposition 9.5 *If* $(\underline{Q}, \underline{W})$ *is a self-injective quiver with potential constructed from a symmetric Postnikov diagram, then* $(\underline{Q}, \underline{W})$ *has enough cuts.*

Proof The planar embedding of $\underline{Q}$ can be taken to be a so-called *isoradial embedding* [4, Theorem 5.7]. This means that all the faces (i.e. cycles in $\underline{W}$) of $\underline{Q}$ are polygons inscribed in a unit circle. Then proceed as follows. Pick an arrow a, and a face F adjacent to a. Without loss of generality, assume that F is oriented clockwise. Now choose a point on the unit circle lying on the arc determined by a on the circle around F. Mark the same point on every copy of the unit circle around all clockwise-oriented faces. One can make the initial choice of a point such that no vertices are marked this way. For every clockwise-oriented face F', mark the arrow on its boundary corresponding to the arc determined by the marked point on the circle around F'. The set of arrows marked this way has the following property: every face has exactly one boundary arrow in this set, except possibly some counterclockwise-oriented faces adjacent to the boundary of the quiver [5, §0.9]. Thus if we choose one boundary arrow for each of these faces, we get a cut containing a and we are done. □

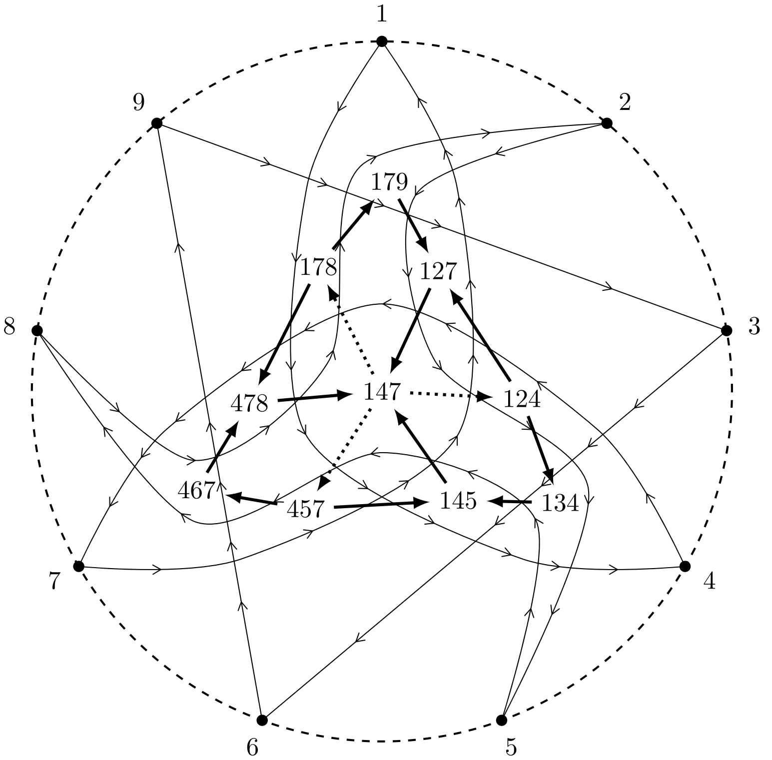

Fig. 11 The cut-mutation $\mu^+_{457}(C)$ of the cut C of Fig. 9

Corollary 9.6 *If $(\underline{Q}, \underline{W})$ is a self-injective quiver with potential constructed from a symmetric Postnikov diagram, then all truncated Jacobian algebras $\wp(\underline{Q}, \underline{W})_C$ are iterated 2-APR tilts of each other. In particular they are derived equivalent.*

Thus we get not only a way of generating many new examples of self-injective quivers with potential, but also the corresponding new 2-representation finite algebras.

An interesting property that 2-representation finite algebras can have is that of being *l-homogeneous* (see [8, Definition 1.2]). It follows from [8, Theorem 2.3] that a truncated Jacobian algebra $\wp(R, W)_C$ as above is l-homogenous for some l if and only if $\psi(C) = C$ (which in our case just means that C is invariant under rotation by $2\pi k/n$). Thus, examples coming from Postnikov diagrams are a good source of l-homogeneous, 2-representation finite algebras. One property that these algebras have is the following:

Theorem 9.7 [8, Theorem 1.3] *A finite-dimensional algebra of global dimension at most 2 is l-homogeneous 2-representation finite if and only if it is twisted $2\frac{l-1}{l}$-Calabi-Yau.*

Twisted fractionally Calabi-Yau algebras can be tensored over $\mathbb{C}$, so we get for instance:

Proposition 9.8 [8, Corollary 1.5] *If Λ_1, Λ_2 are l-homogeneous 2-representation finite algebras, then $\Lambda_1 \otimes_{\mathbb{C}} \Lambda_2$ is l-homogeneous 4-representation finite.*

As we mentioned, in the case of Postnikov diagrams it is easy to see whether a truncated Jacobian algebra is homogeneous: one needs to check whether the cut is invariant under ρ. For instance, the truncated Jacobian algebra of Fig. 9 is not homogeneous, but the one of Fig. 11 is. In particular, the N-fold tensor product of the latter algebra with itself is $2N$-representation finite.

10 Examples

We present some self-injective quivers with potential obtained from symmetric (k, n)-Postnikov diagrams. The Nakayama permutation acts by rotation by $2\pi k/n$, and has order $a = n \big/ \mathrm{GCD}(k, n)$.

The quivers with potential on the left hand side of Fig. 19 and of Fig. 26 (corresponding to $(k, n) \in \{(3, 12), (4, 10)\}$) had already been found by Martin Herschend, and the latter had also been found independently by Sefi Ladkani. These results are not published. A symmetric (4, 8)-Postnikov diagram had appeared in [13, Section 11].

10.1 The Case $a = 2$

If $a = 2$ then we must have $n = 2k$.

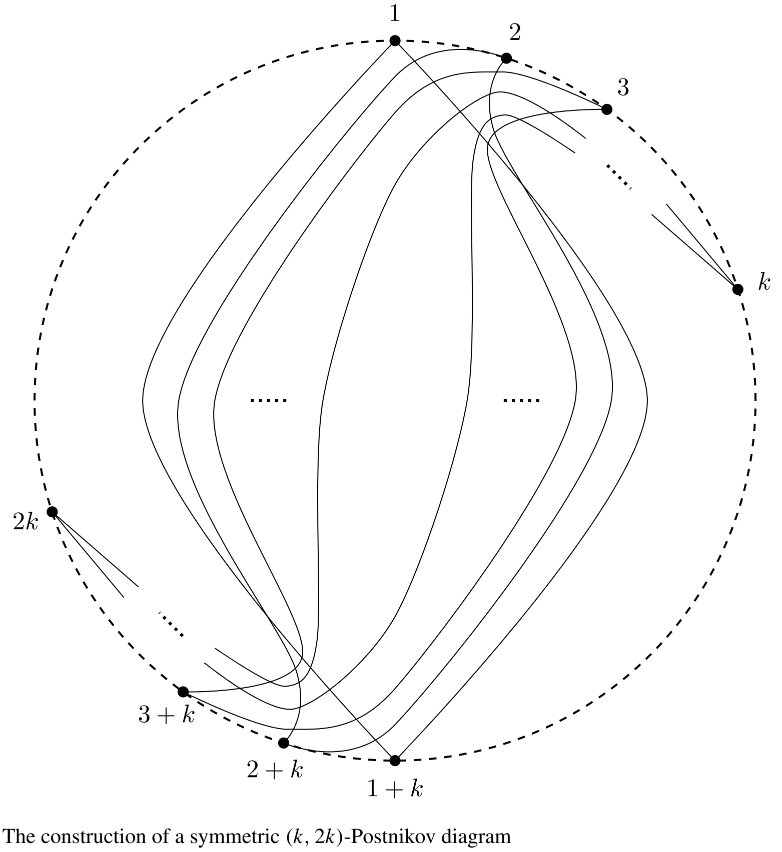

Fig. 12 The construction of a symmetric $(k, 2k)$-Postnikov diagram

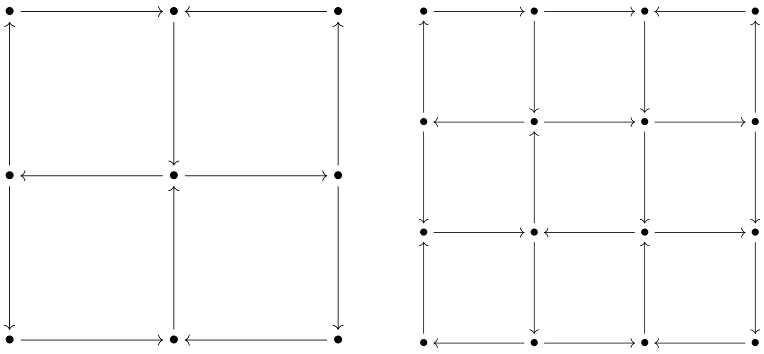

Fig. 13 Two square grid self-injective quivers with potential (here $n = 2k$, $k = 4, 5$)

Proposition 10.1 *For every $k > 1$, there exists a symmetric $(k, 2k)$-Postnikov diagram whose associated self-injective quiver with potential is a square grid with $(k-1)$ vertices on each side.*

Proof The construction in Fig. 12 yields such a symmetric Postnikov diagram, and it produces the correct quiver. To avoid clogging the picture, we have not marked the direction of the strands. They should be understood as follows: strand i crosses strand $i+k$ coming from the left at vertex i if and only if i is odd. The strands k and $2k$ cross strands $k-1, 2k-2, k-3, 2k-4, \ldots, k-2, 2k-1$ in this order for k and the opposite for $2k$, or viceversa depending on the parity of k. $\square$

Such quivers and their planar mutations were already studied in [9, §9.3]. In Fig. 13 we show the cases $k = 4, 5$.

10.2 The Case $a = 3$

If $a = 3$ then we may assume $n = 3k$. Notice that we are treating the cases of clockwise and counterclockwise rotation together, and this is justified by the fact that now we are focusing on the self-injective algebra, which does not change if we reflect the quiver (even though the two quivers are not isomorphic as planar quivers with faces). Here one could expect to get the family of self-injective quivers with potential given by 3-preprojective algebras of type A_j (cf. [9, §9.2]). This is true for $k \in \{2, 3, 4\}$, where we get the quivers in Figs. 14 and 15

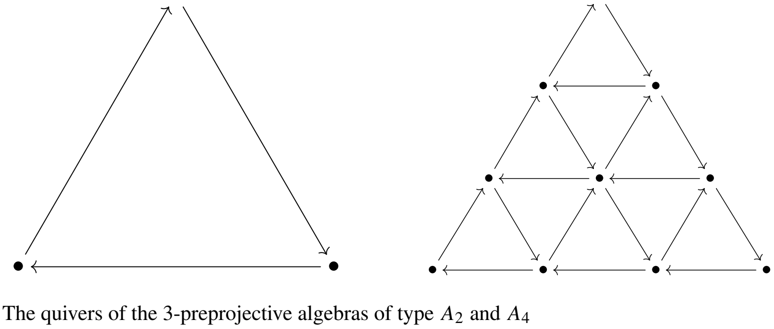

Fig. 14 The quivers of the 3-preprojective algebras of type A_2 and A_4

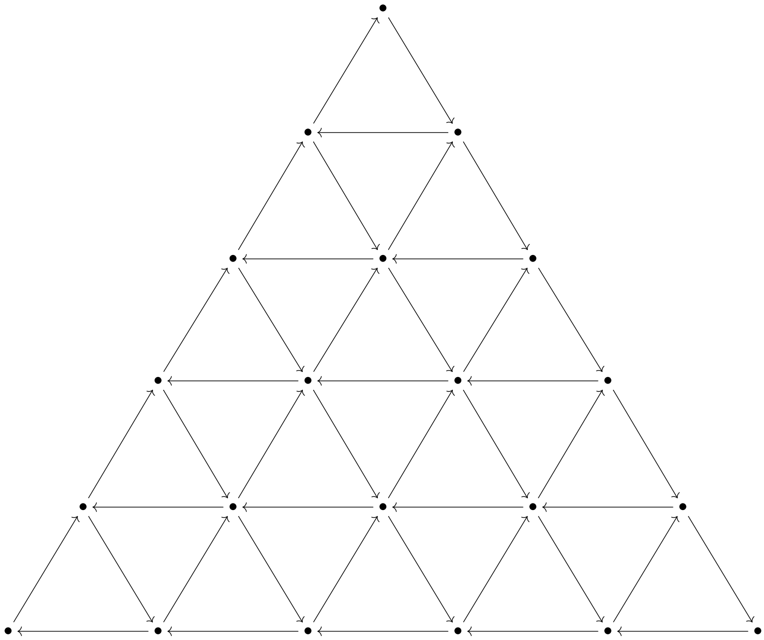

Fig. 15 The quiver of the 3-preprojective algebra of type A_6

(corresponding to type A_2, A_4, A_6). Notice that the quiver corresponding to the symmetric Postnikov diagram of Fig. 2 is equivalent to the one of type A_4 by mutation at the orbit consisting of the vertices of the big triangle. Notice that type A_j with j odd cannot appear this way, since the number of alternating internal faces of a $(k, 3k)$-Postnikov diagram is $(k-1)(2k-1)$. The Postnikov diagrams corresponding to these three quivers are shown in Figs. 16, 17 and 18.

10.3 The Case $a = 4$

For $a = 4$ (we may then assume that $n = 4k$, since the only elements of order 4 in $\mathbb{Z}/4\mathbb{Z}$ are ± 1) we present three self-injective quivers with potential coming from symmetric Postnikov diagrams, for $(k, n) \in \{(3, 12), (4, 16), (5, 20)\}$. They are shown in Figs. 19 and 20. The corresponding Postnikov diagrams are shown in Figs. 21, 22 and 23.

10.4 Cobwebs

We obtain a new infinite family of self-injective quivers with potential, with arbitrarily large order of σ.

For any odd integer $x \geq 3$, define a graph $\mathrm{Cob}(x)$ as follows. Set

$$\mathrm{Cob}(x)_0 = \{c_1, \ldots, c_x\} \cup \{d_{st}\}_{s=1,\ldots,(x-3)/2}^{t=1,\ldots,2x}.$$

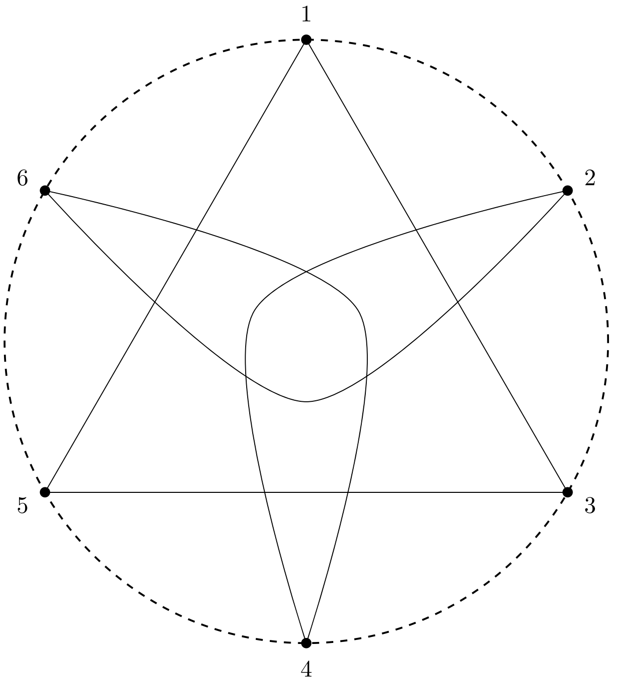

Fig. 16 The (2, 6)-Postnikov diagram corresponding to the 3-preprojective algebra of type A_2

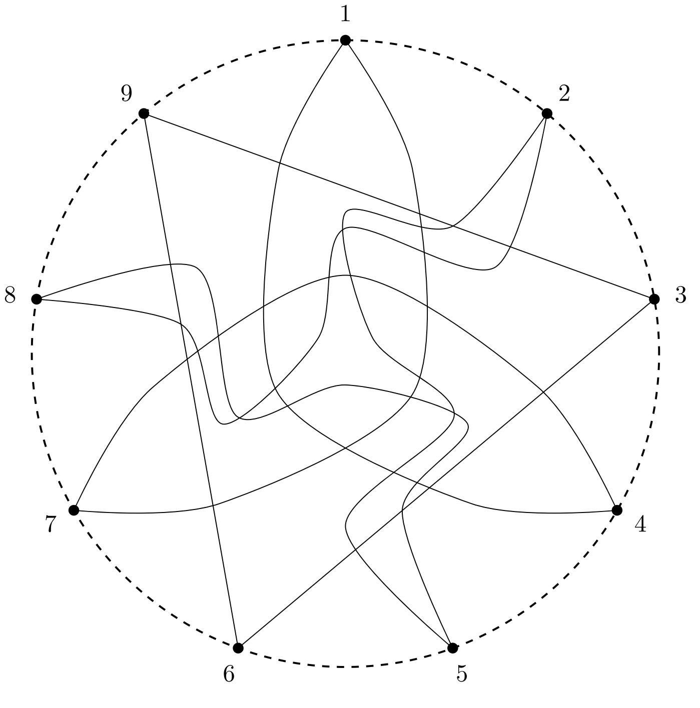

Fig. 17 The (3, 9)-Postnikov diagram corresponding to the 3-preprojective algebra of type A_4

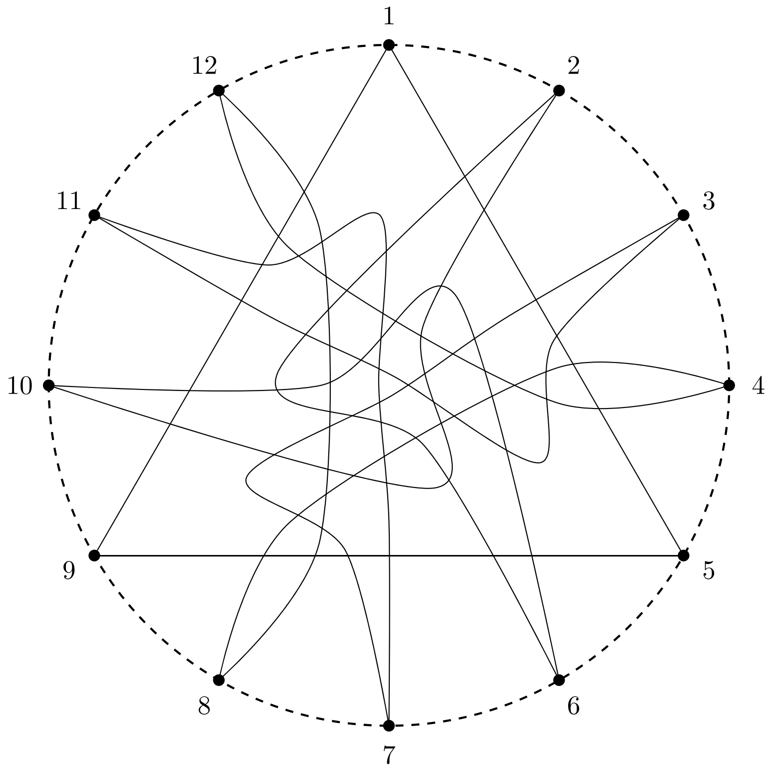

Fig. 18 The (4, 12)-Postnikov diagram corresponding to the 3-preprojective algebra of type A_6

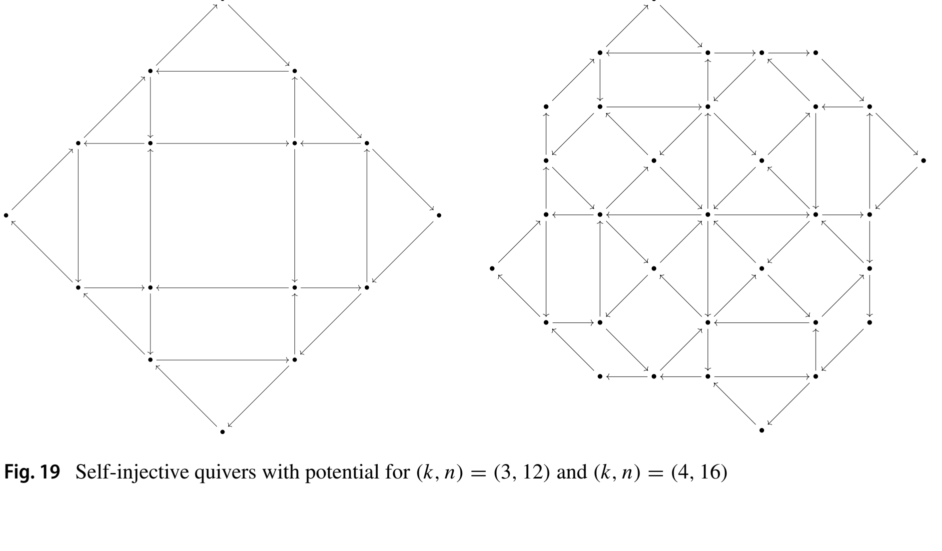

Fig. 19 Self-injective quivers with potential for $(k, n) = (3, 12)$ and $(k, n) = (4, 16)$

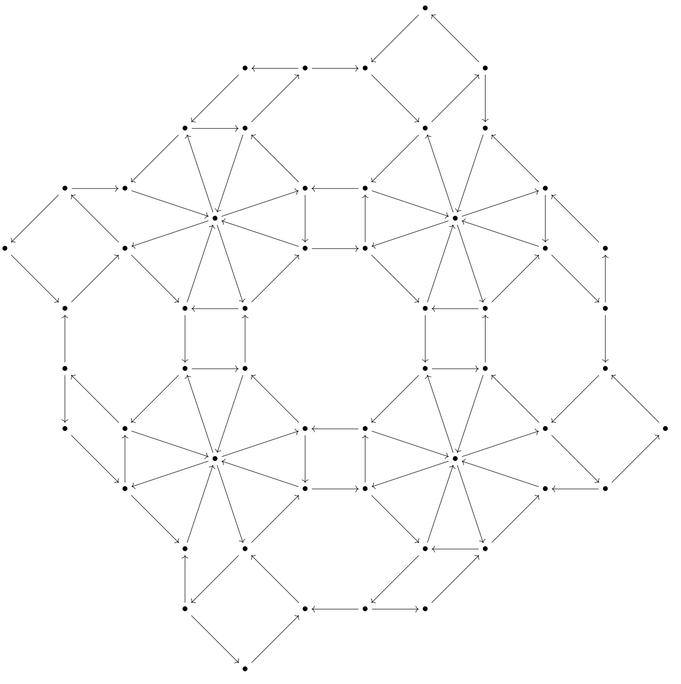

Fig. 20 A self-injective quiver with potential for $(k, n) = (5, 20)$

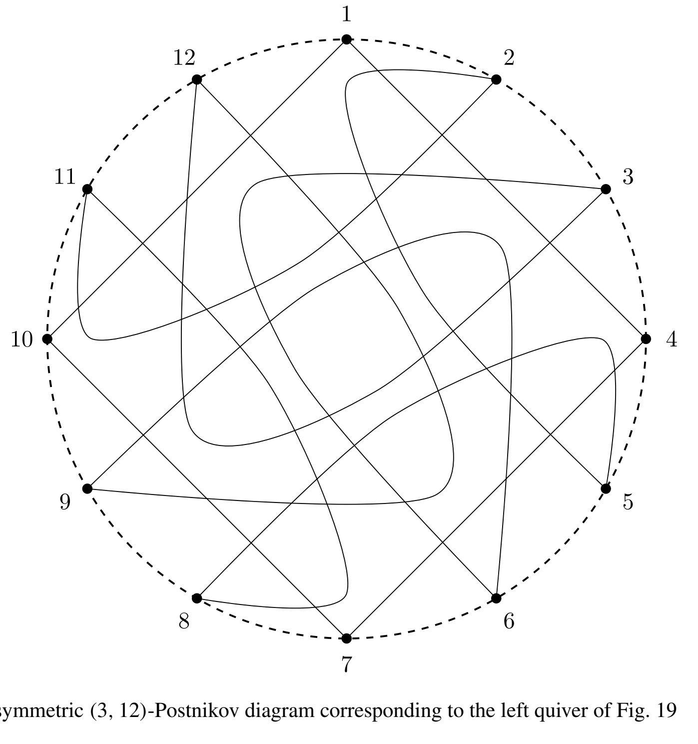

Fig. 21 The symmetric (3, 12)-Postnikov diagram corresponding to the left quiver of Fig. 19

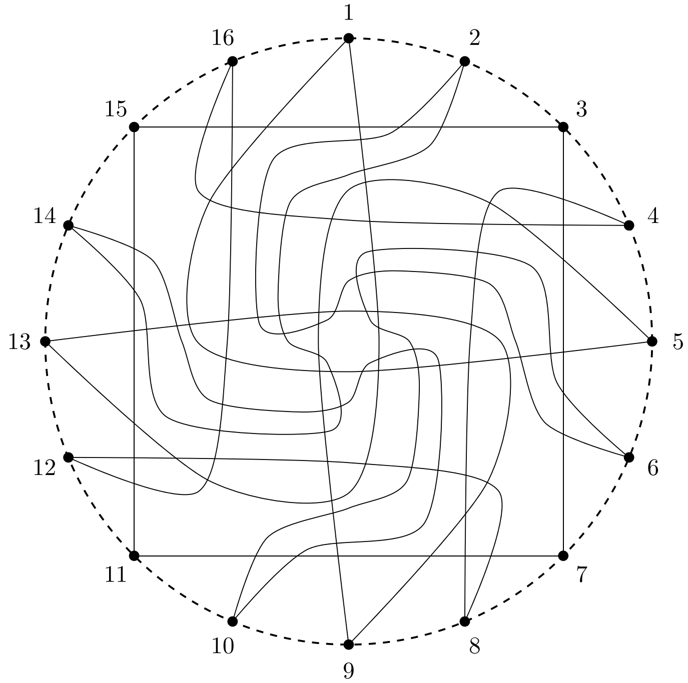

Fig. 22 The symmetric (4, 16)-Postnikov diagram corresponding to the right quiver of Fig. 19

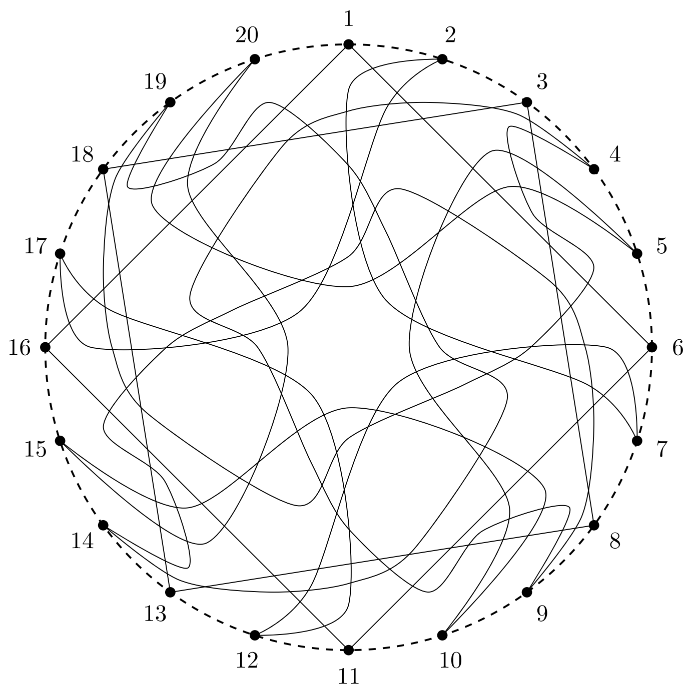

Fig. 23 The symmetric (5, 20)-Postnikov diagram corresponding to the quiver of Fig. 20

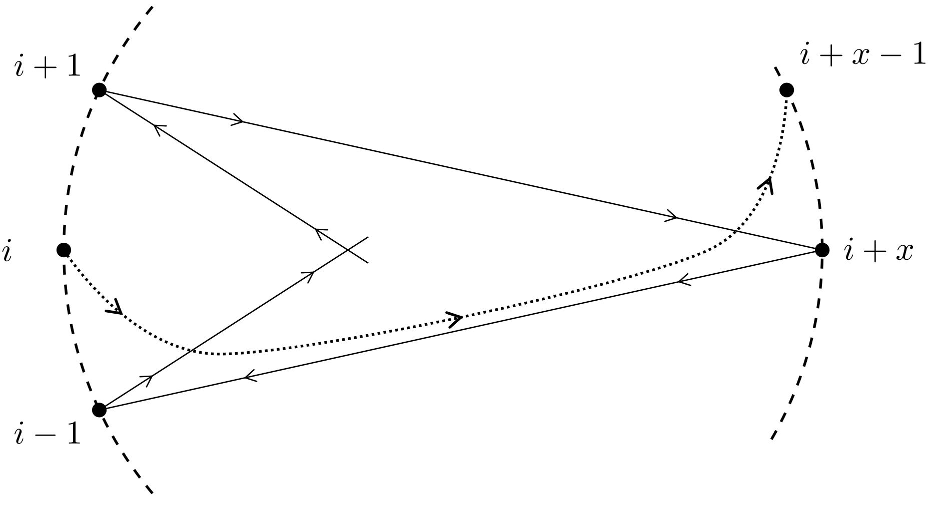

Fig. 24 How to draw strand i for i even in $(x-1, 2x)$-Postnikov diagrams

Set

$$\begin{aligned}\mathrm{Cob}(x)_1 &= \{(c_t, c_{t+1})\}_{t=1,\dots,x}\\ &\cup \left\{(d_{st}, d_{s,t+1})\right\}_{s=1,\dots,(x-3)/2}^{t=1,\dots,2x}\\ &\cup \left\{(d_{st}, d_{s+1,t})\right\}_{s=1,\dots,(x-5)/2}^{t=1,\dots,2x}\\ &\cup \left\{(c_t, d_{1,2t-1}), (c_t, d_{1,2t})\right\}_{t=1,\dots,x}\end{aligned}$$

where indices are taken modulo x in the first row and modulo $2x$ in the second row. This graph has a natural embedding in the plane given by arranging the vertices c_t clockwise in a regular x-gon of radius 1, and the vertices d_{st} clockwise in a regular $2x$-gon of radius $s+1$ for every s. This embedding equips $\mathrm{Cob}(x)$ with faces bounded by cycles (one x-gon, x triangles and x^2-4x squares). Choosing an orientation of an edge, we can turn $\mathrm{Cob}(x)$ into a quiver by requiring that all these cycles be cyclically oriented. Call $\mathrm{Cob}^+(x)$ and $\mathrm{Cob}^-(x)$ the quivers one gets by orienting the x-gon counterclockwise and clockwise respectively (see Figs. 26 and 27). As usual, one can define potentials on these quivers by taking the alternating sum of all cycles bounding faces.

Proposition 10.2 *For every odd* $x \geq 3$*, there exists a symmetric* $(x-1, 2x)$*-Postnikov diagram whose associated self-injective quiver with potential is* $\mathrm{Cob}^+(x)$*. Similarly, there exists a symmetric* $(x+1, 2x)$*-Postnikov diagram whose associated self-injective quiver with potential is* $\mathrm{Cob}^-(x)$*.*

Proof We give the construction for the first case, the second case being similar. Start by connecting vertex i to $i+x-1$ with a straight strand for every i odd (creating a x-pointed star shape). Then for every i even, draw a strand $i \to i+x-1$ as in Fig. 24: cross strand $i-1$, then follow strand $i+x$ as close as possible until its start, and cross strand $i+1$ as last crossing. This construction yields a symmetric Postnikov diagram, and it produces the correct quiver. $\square$

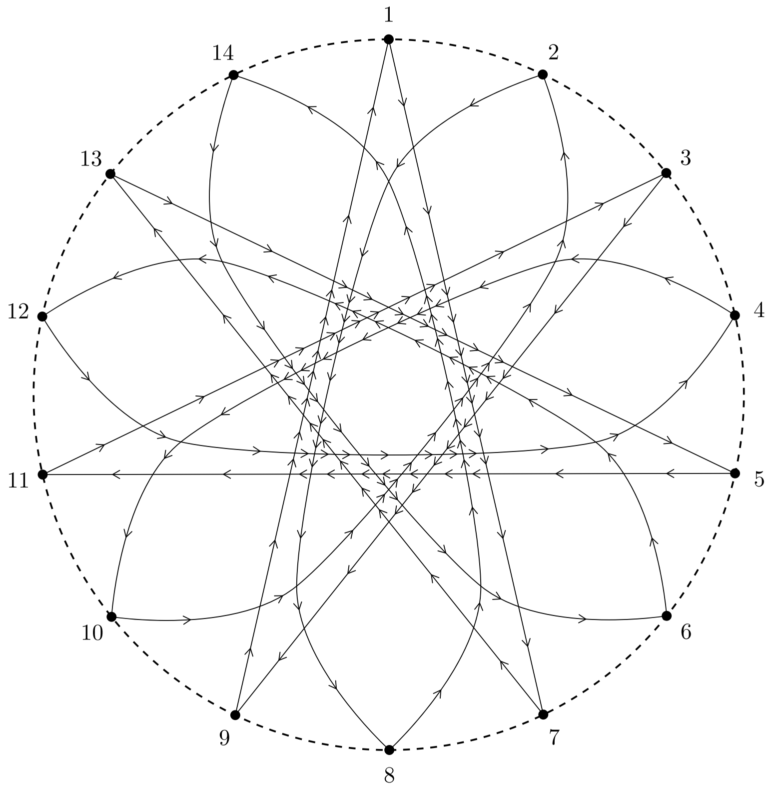

Fig. 25 A symmetric (6, 14)-Postnikov diagram

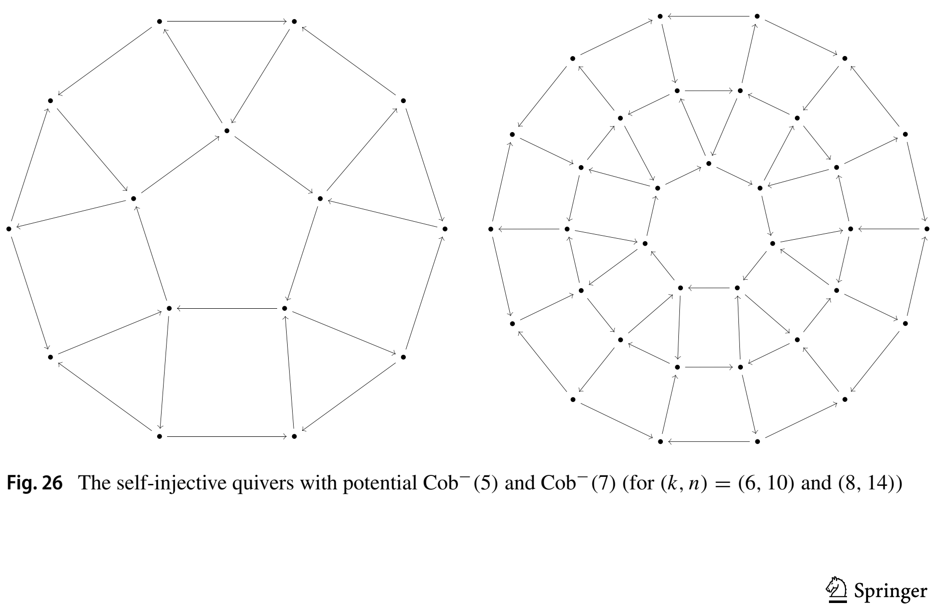

Fig. 26 The self-injective quivers with potential $\mathrm{Cob}^{-}(5)$ and $\mathrm{Cob}^{-}(7)$ (for $(k, n) = (6, 10)$ and $(8, 14)$)

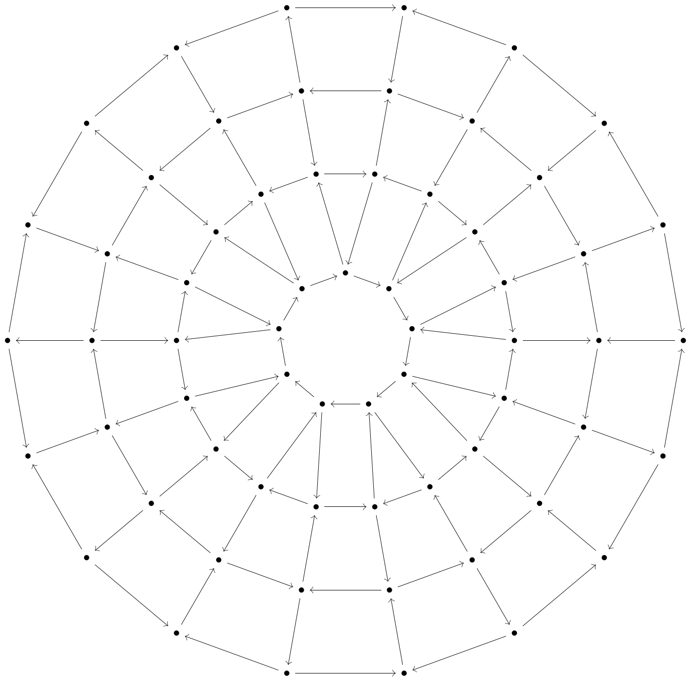

Fig. 27 The self-injective quiver with potential $\mathrm{Cob}^{-}(9)$ (for $(k, n) = (10, 18)$)

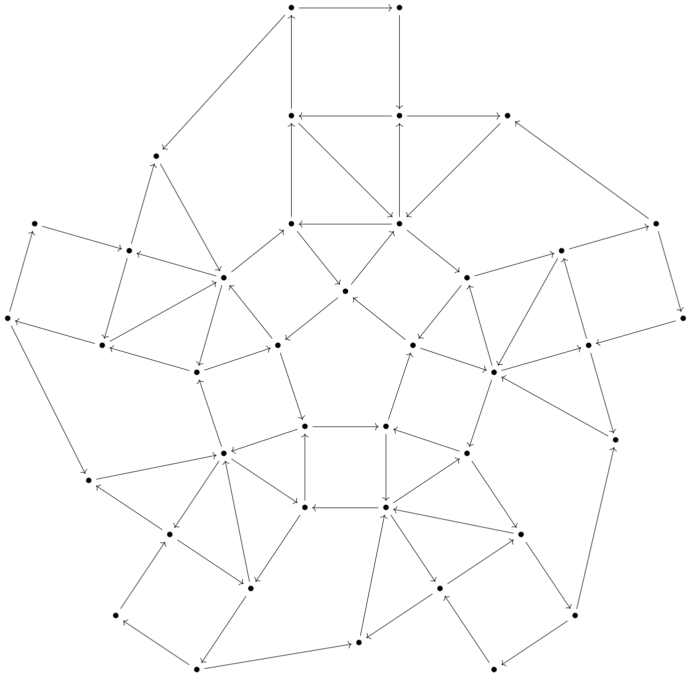

Fig. 28 A self-injective quiver with potential for $(k, n) = (6, 15)$

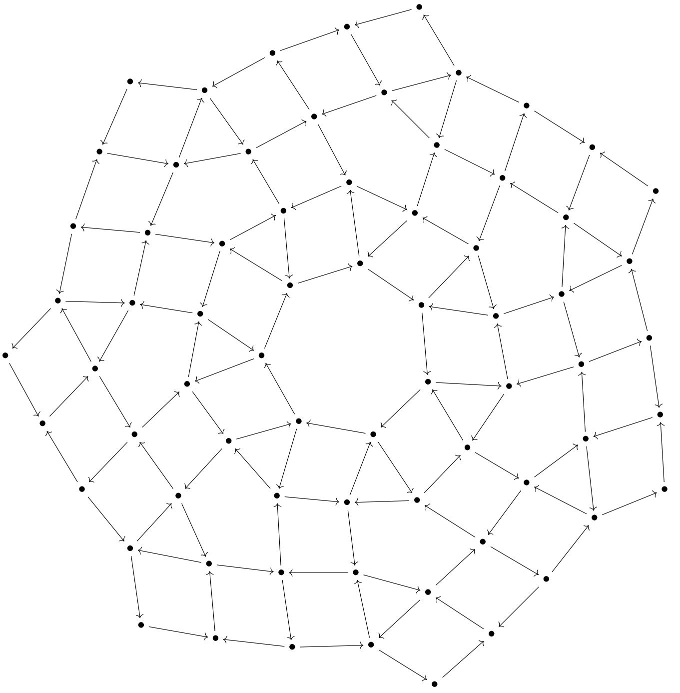

Fig. 29 A self-injective quiver with potential for $(k, n) = (6, 21)$

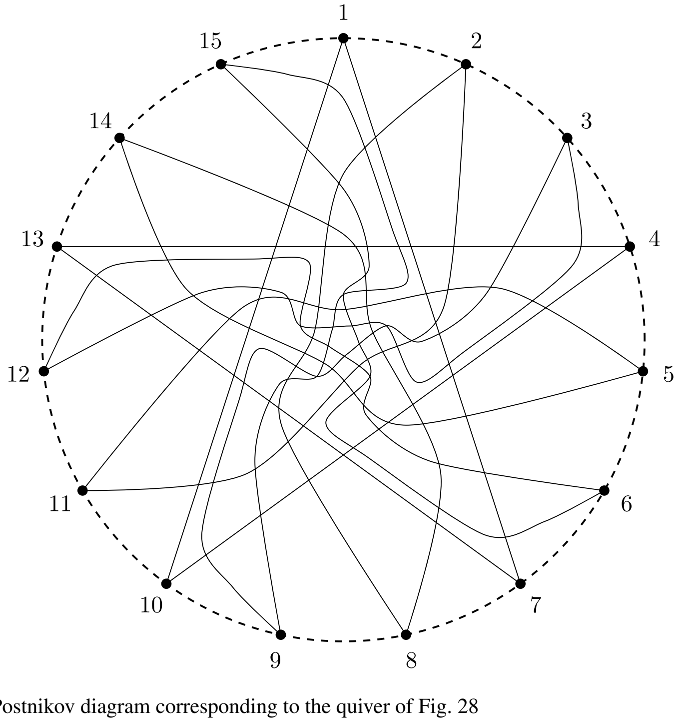

Fig. 30 The Postnikov diagram corresponding to the quiver of Fig. 28

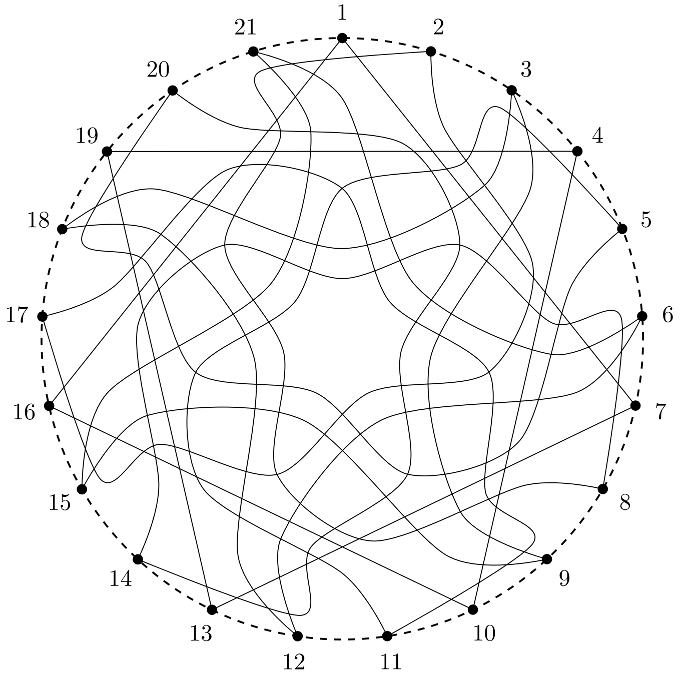

Fig. 31 The Postnikov diagram corresponding to the quiver of Fig. 29

In Fig. 25 we illustrate the case $x = 7$. The quivers $\mathrm{Cob}^{-}(x)$ are shown in Figs. 26 and 27 for $x = 5, 7, 9$. The Nakayama permutation acts by rotation by $\pi(x+1)/x$, which has order x.

10.5 Miscellaneous

We have two more examples of self-injective quivers with potential coming from symmetric Postnikov diagrams, for $(k, n) = (6, 15)$ and $(6, 21)$. We show the first one in Fig. 28. For $(k, n) = (6, 21)$ we get Fig. 29. In Figs. 30 and 31 we show the corresponding Postnikov diagrams.

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Paper IV

EXISTENCE OF SYMMETRIC MAXIMAL NONCROSSING COLLECTIONS OF k-ELEMENT SETS

ANDREA PASQUALI, ERIK THÖRNBLAD, AND JAKOB ZIMMERMANN

ABSTRACT. We investigate the existence of maximal collections of mutually noncrossing k-element subsets of $\{1, \ldots, n\}$ that are invariant under adding $k \pmod n$ to all indices. Our main result is that such a collection exists if and only if k is congruent to $0, 1$ or -1 modulo $n/\operatorname{GCD}(k, n)$. Moreover, we present some algebraic consequences of our result related to self-injective Jacobian algebras.

INTRODUCTION

Two subsets I and J of $\{1, \ldots, n\}$ are said to be *noncrossing* if there are no cyclically ordered a, b, c, d such that $a, c \in I \setminus J$ and $b, d \in J \setminus I$. We are interested in maximal collections of mutually noncrossing sets of some fixed size k. In the case $k = 2$, such collections are maximal collections of noncrossing segments between n points on a circle, that is, triangulations of the n-gon. The case of general k can be tackled using the machinery of alternating strand diagrams and plabic graphs developed by Postnikov [Pos06].

We call a collection of k-element subsets of $\{1, \ldots, n\}$ *symmetric* if it is invariant under adding k $\pmod n$ to all indices. In a recent paper [Pas17] Pasquali showed that symmetric maximal noncrossing collections naturally give rise to self-injective Jacobian algebras. More precisely he showed that any maximal symmetric noncrossing collection for a pair (k, n) gives rise to a self-injective Jacobian algebra whose quiver with potential can be obtained from an embedding of the collection into the plane. Now a natural question is for which pairs (k, n) does there exists such a collection. In this paper we answer this question by the following theorem.

Theorem (Theorem 1.6)**.** *Let $(k, n) \in \mathbb{Z}^2$, with $n \geq 1$ and $0 \leq k \leq n$, and call $d = n/\operatorname{GCD}(k, n)$. Then the following are equivalent:*

- *there exists a symmetric maximal noncrossing collection of k-element subsets of $\{1, \ldots, n\}$;*
- *the number k is congruent to $0, 1$ or -1 modulo d.*

Our proof goes via an explicit construction of a symmetric maximal noncrossing collection. It should be noted that there exist symmetric maximal noncrossing collections which do not arise in this way. In particular the problem of classifying all symmetric maximal noncrossing collections is, as far as we know, still open.

Our motivation for studying maximal noncrossing collections comes as mentioned above from algebra, specifically representation theory and cluster theory. In the following we give a more detailed account of the connection between the combinatorics of maximal noncrossing collections and algebra. It turns out that the combinatorics of noncrossing sets corresponds to the cluster combinatorics of the homogeneous coordinate ring of the Grassmannian $\operatorname{Gr}_{\mathbb{C}}(k, n)$; see [Sco06]. Every cluster consisting of Plücker coordinates corresponds to a maximal noncrossing collection, and both the cluster variables and the quiver corresponding to the cluster can be constructed from the collection (see [OPS15]). Moreover, there is a categorification of this cluster structure using Cohen-Macaulay modules over an infinite dimensional algebra $B = B(k, n)$ [JKS16]. An indecomposable Cohen-Macaulay B-module is associated to every k-element subset of $\{1, \ldots, n\}$, and the noncrossing condition corresponds to the vanishing of Ext^1_B between these modules. Thus a maximal noncrossing collection corresponds to a cluster tilting object in the category $\operatorname{CM}(B)$ of Cohen-Macaulay B-modules. It was shown in [BKM16] that in fact the endomorphism ring of this cluster tilting module is a frozen Jacobian algebra. The cluster of Plücker coordinates corresponding to the collection gives the quiver with potential of this algebra.

One can also consider the analogous story in the stable category $\underline{\operatorname{CM}}(B)$, which corresponds to taking a quotient by the idempotent corresponding to the frozen vertices. Then one gets a finite-dimensional Jacobian algebra Λ, and using the triangulated structure of $\underline{\operatorname{CM}}(B)$ one can prove that Λ is self-injective if and only if the corresponding maximal noncrossing collection is invariant under adding k to all indices $\pmod n$. The operation of adding k modulo n is nothing but a planar rotation of the quiver (or of the boundary n-gon) by an angle of $\frac{2\pi k}{n}$. One reason to look at self-injective Jacobian algebras is that they are precisely the 3-preprojective algebras of 2-representation finite algebras [HI11].

These algebras have an automorphism called the Nakayama automorphism. One can consider the order of this automorphism (when it is finite), and all known examples have either small order or are in some sense very simple (roughly speaking, the number of vertices is linear in the order). A consequence of our result is:

Corollary (Corollary 7.3)**.** *Let $d \in \mathbb{Z}_{>1}$. There exist infinitely many self-injective Jacobian algebra with a Nakayama automorphism of order d.*

The paper is organised in the following way. The next section introduces the problem and gives the statement of our main theorem. In Section 2 we prove one direction of Theorem 1.6, using planar embeddings of noncrossing collections. Section 3 is devoted to setting up the notation for our constructions and proving some auxiliary results. In Sections 4 and 5 we construct an explicit symmetric maximal noncrossing collection, first if $k \mid n$ and then for general (k, n). This provides the other direction in the proof of Theorem 1.6. In Section 6 we compute an explicit example to illustrate our construction. Finally, in Section 7 we explain some algebraic consequences of our combinatorial result, in particular about existence of self-injective Jacobian algebras.

Acknowledgements. We would like to express our gratitude to Martin Herschend and Laertis Vaso, for carefully reading the manuscript and providing helpful feedback.

1. The problem

In the following let $0 < k \leq n$ be integers. A *cyclically ordered set* is a finite set X together with a bijection $S_X : X \to X$ such that for all $x, y \in X$ there is $n \in \mathbb{Z}$ such that $S_X^n(x) = y$. We think of S_X as a "successor function". If X is a cyclically ordered set and $\varnothing \neq Q \subseteq X$, there is an induced cyclic order S_Q on Q. Indeed, for $q \in Q$, we define $S_Q(q) = S_X^m(q)$, where $m > 0$ is the least positive integer such that $S_X^m(q) \in Q$. If X is cyclically ordered and $x_1, x_2, \dots, x_n \in X$ are distinct elements, we write

$$x_1 <_\circ x_2 <_\circ \cdots <_\circ x_n$$

if $S_Q(x_i) = x_{i+1}$ for $1 \leq i < n$, where $Q = \{x_1, x_2, \dots, x_n\}$. With this we can now give the following definition:

Definition 1.1. Let X be a cyclically ordered set. Two subsets $I, J \subseteq X$ are said to be *crossing* if there exist $a <_\circ b <_\circ c <_\circ d \in X$ such that $a, c \in I \setminus J$ and $b, d \in J \setminus I$. Otherwise I, J are said to be *noncrossing*.

For $n \in \mathbb{N}$, we write $[n] = \{1, 2, \dots, n\}$. For $a, b \in [n]$, we write

$$a \overset{n}{+} b = \begin{cases} a + b, & \text{if } a + b \in [n]; \\ a + b - n, & \text{otherwise.} \end{cases}$$

We will always consider the cyclic order on $[n]$ given by $S_{[n]}(x) = x \overset{n}{+} 1$. For a subset I of $[n]$, we denote by $I \overset{n}{+} k$ the subset $\left\{ i \overset{n}{+} k \mid i \in I \right\}$ of $[n]$. For a collection $\mathbb{I}$ of subsets of $[n]$, we denote by $\mathbb{I} \overset{n}{+} k$ the collection $\left\{ I \overset{n}{+} k \mid I \in \mathbb{I} \right\}$.

Definition 1.2. A collection $\mathbb{I}$ of k-element subsets of $[n]$ is a (k, n)*-noncrossing collection* if I and J are noncrossing for all $I, J \in \mathbb{I}$. A (k, n)-noncrossing collection is *maximal* if it is maximal with respect to inclusion in the set of all (k, n)-noncrossing collections.

We will often omit the reference to (k, n) when it is clear from the context.

Definition 1.3. A collection $\mathbb{I}$ of k-element subsets of $[n]$ is *symmetric* if $\mathbb{I} = \mathbb{I} \overset{n}{+} k$.

Observe that this definition depends on k and n. In case these are ambiguous we will spell out $\mathbb{I} = \mathbb{I} \overset{n}{+} k$.

We can now formally state the question which we address in this paper.

Question 1. *For which (k, n) does there exist a maximal (k, n)-noncrossing collection which is symmetric?*

Remark 1.4. It is worth pointing out that we look for symmetric collections which are maximal among all collections, not just among the symmetric ones.

It is easy to see that such collections do not exist for all choices of n and k. It turns out that the following condition is what we need.

Condition 1.5. *The pair $(k,n) \in \mathbb{Z}^2$ is such that $n \geq 1$, $0 \leq k \leq n$, and k is congruent to $0, 1$ or -1 modulo $n/\operatorname{GCD}(k,n)$.*

Indeed, we have the following:

Theorem 1.6. *There exists a symmetric maximal (k,n)-noncrossing collection if and only if (k,n) satisfies Condition 1.5.*

We give a proof of Theorem 1.6 at the end of Section 5. The strategy is as follows: in Section 2 we will prove that Condition 1.5 is necessary, and in Sections 4 and 5 we will explicitly construct a symmetric maximal noncrossing collection to show that Condition 1.5 is sufficient.

Remark 1.7. Observe that $I, J \subseteq [n]$ are noncrossing if and only if the complements $[n] \setminus I$ and $[n] \setminus J$ are. Moreover, a pair (k,n) satisfies Condition 1.5 if and only if the pair $(n-k,n)$ does. Thus it is enough to study the case $k \leq \frac{n}{2}$. Our construction and result work for general $k \leq n$, so we do not make this assumption.

To prove Theorem 5.4 we will in fact use a characterisation of maximal noncrossing collections, which was first conjectured in [Sco05] (see also [LZ98]) and then proved in [OPS15].

Theorem 1.8 ([OPS15, Theorem 4.7])**.** *A (k,n)-noncrossing collection $\mathbb{I}$ of k-element subsets of $[n]$ is a maximal (k,n)-noncrossing collection if and only if $|\mathbb{I}| = k(n-k)+1$.*

Remark 1.9. This implies that Question 1 is equivalent to: for which (k,n) does there exist a symmetric noncrossing collection of cardinality $k(n-k)+1$? In Sections 4 and 5 we will construct collections of noncrossing sets and prove that they are maximal by determining that they have the correct cardinality.

2. Necessity of Condition 1.5

The aim of this section is to prove the following statement:

Proposition 2.1. *If there exists a symmetric maximal (k,n)-noncrossing collection, then (k,n) must satisfy Condition 1.5.*

The proof uses combinatorial tools developed in [OPS15], in particular a planar CW-complex which is associated to a noncrossing collection. We recall some details about its construction for convenience.

Let $v_1, \ldots, v_n$ be the vertices of a regular n-gon in $\mathbb{R}^2$ centered at the origin, labeled in clockwise order. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$, and define a linear map $p : \mathbb{R}^n \to \mathbb{R}^2$ by $p(e_i) = v_i$ for all i. If I is any subset of $[n]$, we define $e_I = \sum_{i \in I} e_i \in \mathbb{R}^n$.

Let now $\mathbb{I}$ be a maximal noncrossing (k,n)-collection. Denote by V the set $\{e_I \mid I \in \mathbb{I}\} \subseteq \mathbb{R}^n$, then we have $p(e_I) = \sum_{i \in I} v_i$. This defines an embedding of $\mathbb{I}$ as a discrete collection of points in $\mathbb{R}^2$. There is a way of defining edges and faces so that we get a CW-complex $\Sigma(\mathbb{I})$, which is also embedded in $\mathbb{R}^2$ and has $p(\mathbb{I})$ as its set of vertices [OPS15, Proposition 9.4]. Faces of $\Sigma(\mathbb{I})$ are parametrised by some subsets of $[n]$ of cardinality $k-1$ and $k+1$. Every edge is in the boundary of two faces, one corresponding to $k-1$ and one to $k+1$. Moreover, we have that $\Sigma(\mathbb{I})$ is homeomorphic to a disk [OPS15, Theorem 9.12 and Theorem 11.1].

We will need the following lemma.

Lemma 2.2. *Let I be a subset of $[n]$ of size l, and assume that $I = I \overset{n}{+} k$. Then $d \mid l$, where $d = n/\operatorname{GCD}(k,n)$.*

Proof. Observe that d is the order of the function $\varphi : a \mapsto a + k$ on $\mathbb{Z}/n\mathbb{Z}$. Moreover, every element of $\mathbb{Z}/n\mathbb{Z}$ has the same order under φ. Seeing I as a subset of $\mathbb{Z}/n\mathbb{Z}$, we get then that $\varphi(I) = I$ and so I is a union of φ-orbits. Since all φ-orbits have size d, we get the claim. □

Proof of Proposition 2.1. If $k \in \{0, 1, n-1, n\}$, then Condition 1.5 is satisfied. Therefore, assume in the following that $1 < k < n-1$.

Since $\mathbb{I}$ is symmetric, we know that $\mathbb{I} = \mathbb{I} \overset{n}{+} k$. Denoting by ρ the clockwise rotation by $\frac{2\pi k}{n}$ centered at the origin, we have by definition

$$p(e_{i \overset{n}{+} k}) = v_{i \overset{n}{+} k} = \rho(v_i) = \rho\left(p(e_i)\right).$$

This in turn implies that $\rho(\Sigma(\mathbb{I})) = \Sigma(\mathbb{I})$. Observe that $d = n/\operatorname{GCD}(k,n)$ is the order of ρ, and that $1 < d < n$. In particular, observe that $\Sigma(\mathbb{I})$ must contain the origin since it is a disk.

Let us look at the origin $\bar{0} \in \mathbb{R}^2$, which is the only fixed point of ρ. Three cases can happen:

(1) $\bar{0} = p(e_I)$ for some $I \in \mathbb{I}$. Then $I = I \overset{n}{+} k$, which implies that d divides $|I| = k$ by Lemma 2.2. So Condition 1.5 is satisfied in this case.
(2) $\bar{0}$ is not a vertex of $\Sigma(\mathbb{I})$, but it lies on an edge. This can only happen if $d = 2$, but in this case the two faces having $\bar{0}$ on their boundary cannot be sent to each other by ρ since their parameters have different cardinalities. So this case does not occur.
(3) $\bar{0}$ is in the interior of a face F of $\Sigma(\mathbb{I})$. In this case we must have that $\rho(F) = F$, which implies that the vertices of F are permuted by ρ. These lie on a circle centered at $p(K)$, where K is the label of F [BKM16, Theorem 5.7(a)]. It follows that $\bar{0}$ must be the center of this circle, so $\bar{0} = p(K)$ and thus $K = K \overset{n}{+} k$ since $\rho(\bar{0}) = \bar{0}$. This implies that d divides $|K|$ by Lemma 2.2. Since $|K|$ is either $k-1$ or $k+1$ being the label of F, Condition 1.5 is satisfied.

So in all possible cases Condition 1.5 holds, hence the claim is proved. □

3. Setup and notation

In this section we fix notation and prove some auxiliary results which we will need later. If X is cyclically ordered and $a, b \in X$, define the set $[a, b] \subseteq X$ to be the smallest subset such that $a, b \in [a, b]$ and $S_X([a, b] \setminus \{b\}) \subseteq [a, b]$. We call a set of this type an *interval* of X. Observe that $[a, b] = X$ if and only if $S_X(b) = a$, and otherwise a and b are uniquely determined by $[a, b]$. If $I \neq X$ we call I a *proper interval*. We will write $[a, b]_X$ instead of $[a, b]$ to specify the set X if needed.

For every $i \in X$, there is an associated linear order $<_i$ on X defined by

$$i <_i S_X(i) <_i S_X^2(i) <_i \cdots <_i S_X^{-1}(i).$$

Observe that $a <_i b <_i c$ implies $a <_\circ b <_\circ c$ for all $a, b, c \in X$. There is a bijection between linear orders on X with this property on the one hand and elements i of X on the other hand. If $I = [a, b] \subseteq X$ is a proper interval, the *linear order associated to* I is the order $<_a$.

Lemma 3.1. *Let I be a proper interval of X, and let $<_I$ be the linear order associated to I. If $a, c \in I$ and $b \in X$ are such that $a <_I b <_I c$, then $b \in I$.*

Proof. Immediate from the definition of interval and of $<_I$. □

Example 3.2. Let $n = 8$. Then $I = \{7, 8, 1, 2, 3\}$ is an interval of $[n]$. The linear order associated to I is $7 < 8 < 1 < 2 < 3 < 4 < 5 < 6$. Let $Q = \{1, 3, 4, 6, 7\}$. Then $I \cap Q = \{7, 1, 3\}$ is an interval of Q.

Lemma 3.3. *Let X be cyclically ordered, and let I be an interval of X. Let $a <_\circ b <_\circ c <_\circ d \in X$ with $a, c \in I$. Then $b \in I$ or $d \in I$.*

Proof. If $I = X$ then $b \in I$ and we are done. Otherwise, let $<_I$ be the linear order associated to I. Assume $a <_I c$. If $b <_I a$, then $b <_\circ a <_\circ c$, contradiction. If $c <_I b$, then $a <_\circ c <_\circ b$, contradiction. So we must have $a <_I b <_I c$ hence $b \in I$. In the same way we obtain that $d \in I$ if we assume that $c <_I a$. □

Lemma 3.4. *Let X be a cyclically ordered set, I an interval of X, and $J \subseteq X$. Then I and J are noncrossing.*

Proof. Since I is an interval, if $a <_\circ b <_\circ c <_\circ d \in X$ with $a, c \in I$, then either d or b are in I by Lemma 3.3. So I and J cannot be crossing. □

4. Construction: the case $n = dk$

In order to prove Theorem 1.6, we will construct a symmetric maximal (k, n)-noncrossing collection whenever (k, n) satisfies Condition 1.5. The construction will be performed in two steps: first we will make the additional assumption that $k \mid n$, and in Section 5 we will show how one can get rid of this assumption.

We first give a construction of a symmetric maximal noncrossing collection when $n = dk$ and (k, dk) satisfies Condition 1.5. Observe that in this case we have $\mathrm{GCD}(k, n) = k$, so $n/\mathrm{GCD}(k, n) = d$. By assumption, $k = dp + c$, with $c \in \{-1, 0, 1\}$. For $a \in [n]$, write $\overline{a} = (a + k\mathbb{Z}) \cap [n]$. Choose a total order $\overline{a_1} < \overline{a_2} < \cdots < \overline{a_k}$ on these congruence classes (for simplicity, we assume that $\{a_1, \ldots, a_k\} = \{1, \ldots, k\}$). We construct collections $\mathcal{L}_s$, for $1 \leq s \leq k - p + 1$, in the following way.

Call $P_s = [n] \setminus \bigcup_{i=1}^{s-1} \overline{a_i}$, considered as a cyclically ordered subset of $[n]$. For $1 \leq h \leq d$, write $P_{s,h} = P_s \setminus \left\{ S_{\overline{a_s}}^m(a_s) \,|\, h \leq m < d \right\}$. For fixed h and $i \in [S_{P_s}(a_s - k), a_s]$, define

$$I(i, h) = \{i, S_{P_{s,h}}(i), \ldots, S_{P_{s,h}}^{k-1}(i)\}.$$

We are interested in k-element sets, and it is easy to see that $|I(i,h)| = k$ if and only if $|P_{s,h}| \geq k$ (in particular, the cardinality of $I(i,h)$ is independent of i). However, we note that different choices of h may give rise to the same set $I(i,h)$. In particular it is clear that if P_s is big (in comparison to k), then large values of h will give the same set $I(i,h)$ for fixed i. Therefore, for given values of i and h we set h^* to be the minimal h' for which $I(i,h') = I(i,h)$. This element h^* can be explicitly determined: we have that $h^* = |\overline{a_s} \cap I(i,h)|$, and that h^* is the unique h' such that $S_{P_{s,h}}^{h'-1}(a_s) \in I(i,h)$.

Let B_s be the collection defined by

$$B_s = \{I(i,h) \,|\, i \in [S_{P_s}(a_s - k), a_s],\ 1 \leq h \leq d,\ |I(i,h)| = k\}\,.$$

We define $\mathcal{L}_s = \left\{ I \overset{dk}{+} xk \,|\, I \in B_s, x \in \mathbb{Z} \right\}$ and $\mathbb{I} = \bigcup_{s=1}^{k-p+1} \mathcal{L}_s$.

Remark 4.1. There is a way of generating all the elements of B_s, which we explain informally. We start with $\{a_s\}$. We keep adding successors in P_s until we have a k-element set I (which is an interval of P_s). This is our first set in B_s. If it contains an element in $\overline{a_s}$ which is not a_s, then we can generate another set in B_s by removing the last (in the order $<_{a_s}$ of $\overline{a_s}$) such element, and adding another element at the end of I. However, we cannot add an element of $\overline{a_s}$ in this way, so if the next element we would add is in $\overline{a_s}$ we skip it and add the next one instead. Thus we get another set in B_s.

Now if the latest set we constructed still has an element in $\overline{a_s}$ which is not a_s, we can remove the last such and add another element at the end, and thus produce another set in B_s. We can repeat this until we get a set I such that $I \cap \overline{a_s} = \{a_s\}$.

Now we can start the whole construction again, beginning with $\left\{S_{P_s}^{-1}(a_s), a_s\right\}$. We add successors until we have k elements, and then we modify our resulting set by removing elements in $\overline{a_s}$ and adding elements at the end. We get some more sets in B_s in this way. Repeating this, starting with the various intervals $[S_{P_s}^{-x}(a_s), a_s]$, we get all the sets in B_s. Observe that the last x we try is $x = k - 1$.

Example 4.2. Let us illustrate the construction with a concrete example. Let $(k,n) = (7,28)$, so that $d = 4$ and Condition 1.5 is satisfied (this does not play a role here). We fix a total order $\bar{4} < \bar{6} < \bar{7} < \bar{2} < \bar{1} < \bar{3} < \bar{5}$ on the congruence classes modulo 7. We take $s = 4$, so that $a_s = 2$ and $P_s = \{1,2,3,5,8,9,10,12,15,16,17,19,22,23,24,26\}$. In Figure 1 we draw the set P_4 on the circle, with crosses to indicate the elements of $[28] \setminus P_4$. The orbit $\overline{a_s} = \bar{2} = \{2,9,16,23\}$ is highlighted.

The arcs represent the 7 elements of the set B_4, namely

$$\{24,26,1,2,3,5,8\}\,,\{26,1,2,3,5,8,9\}\,,\{26,1,2,3,5,8,10\}\,,\{1,2,3,5,8,9,10\}\,,$$
$$\{1,2,3,5,8,10,12\}\,,\{2,3,5,8,9,10,12\}\,,\{2,3,5,8,10,12,15\}\,.$$

Observe that they all contain 2 and that some of them but not all contain 9. The set $\mathcal{L}_4$ consists of all the shifts of the 7 sets above by multiples of 7 modulo 28, and has thus 28 elements. The reader is invited to pick two sets in B_4 and check that they are noncrossing, and to do the same with two arbitrary sets in $\mathcal{L}_4$.

Our claim is that $\mathbb{I}$ is a symmetric maximal (k,n)-noncrossing collection. To prove this, we will show that it consists of mutually noncrossing sets and that $|\mathbb{I}| = k(n-k)+1$. Thus we will be able to conclude that the claim holds using Theorem 1.8. In this process, the only step that uses Condition 1.5 is checking the cardinality of the last nonempty $\mathcal{L}_s$.

Lemma 4.3. *If $s \neq t$, then $\mathcal{L}_s \cap \mathcal{L}_t = \varnothing$.*

Proof. By symmetry, assume $s < t$. Since $\overline{a_s} \in I$ for every $I \in B_s$, every element in $\mathcal{L}_s$ contains some $a \in \overline{a_s}$. On the other hand, no element in $\mathcal{L}_t$ contains any such a. □

Now we count the number of elements in each collection $\mathcal{L}_s$.

Proposition 4.4.

(1) *For all $s < k - p$, we have $|\mathcal{L}_s| = kd$.*
(2) *If $k \equiv -1$ or $k \equiv 0 \pmod{d}$, then $|\mathcal{L}_{k-p}| = kd$.*
(3) *If $k \equiv 1 \pmod{d}$, then $|\mathcal{L}_{k-p}| = d(k-p)$.*

Proof. We start by remarking that in the cases we consider we have $|\mathcal{L}_s| = d|B_s|$. Indeed, let $I \in B_s$. Then $J = I \cap \overline{a_s}$ is an interval of $\overline{a_s}$. The sets $I \overset{n}{+} kx$ intersect $\overline{a_s}$ in $J \overset{n}{+} kx$, and the only case in which these d intervals of $\overline{a_s}$ are not all distinct is when $J = \overline{a_s}$, that is when I is an interval of P_s. Now, by the same argument, the d intervals $I + kx$ of P_s are all distinct unless $I = P_s$. In particular, if $|\mathcal{L}_s| \neq d|B_s|$, we must have $|P_s| = k$. Since $|P_s| = d(k-s+1)$, this can only happen (assuming Condition 1.5) if

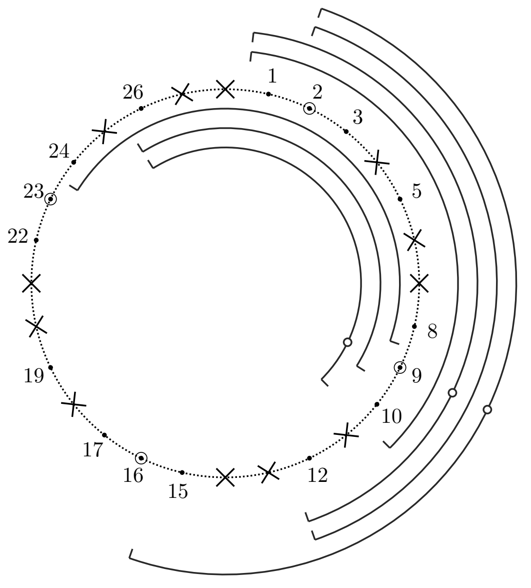

FIGURE 1. An example of our construction for $(k,n) = (7,28)$ and $s = 4$.

$k = dp$ with $p = k - s + 1$. In particular, in the cases we consider in assertions (1)–(3), we must have $|\mathcal{L}_s| = d|B_s|$. As a consequence, it suffices to compute the cardinality of B_s.

To count the elements of B_s, we will identify a suitable domain that makes the function $(i,h) \mapsto I(i,h)$ injective, and count the elements of this domain.

Recall that to to each pair (i,h) we can assign the pair (i,h^*), where h^* is such that $I(i,h) = I(i,h^*)$ and minimal with respect to this property.

On the other hand, if $|P_{s,h}| > k$, we can recover i from the set $I(i,h)$. This is because the set $I(i,h)$ is an interval of $P_{s,h}$, and i is its starting element.

If $|P_{s,h}| = k$, then the sets $I(i,h)$ are all equal to $P_{s,h}$, so these cases will require special attention.

Let $r = k - s + 1$. Observe that there exists a unique bijection that preserves cyclic order from P_s to $[dr]$ such that a_s is sent to r. Therefore we will assume in the following that $P_s = [dr]$.

It will be convenient to fix $i \in [1,r]$ and let h vary. We want to count, for a fixed i, how many sets $I(i,h)$ there are such that $h = h^*$ (to avoid double counting). In other words, how many sets $I(i,h)$ there are such that $rh \in I(i,h)$. The set $I(i,h)$ must contain the interval $[i, rh]$, so from $|I| = k$ we get $k \geq rh - i + 1$. Setting

$$\gamma_i = \left\lfloor \frac{k+i-1}{r} \right\rfloor,$$

we obtain $h \leq \gamma_i$. We conclude that for a fixed $i \in [1,r]$ there are exactly γ_i distinct values of h such that $I(i,h)$ has size k, and thus γ_i elements in B_s.

As we pointed out, one can recover i from $I(i,h)$ unless $|P_{s,h}| = k$, which means that $|B_s| = \sum_{i=1}^{r} \gamma_i$ unless $|P_{s,h}| = k$. Let us analyse the special case $|P_{s,h}| = k$, distinguishing between the three congruences permitted by Condition 1.5.

By construction, $|P_{s,h}| = dk - ds + h$, and recall that $1 \leq h \leq d$. Now there are three cases, assuming $|P_{s,h}| = k$:

- If $k = dp + 1$, then $h = 1$ and $p = k - s$.
- If $k = dp$, then $h = d$ and $p = k - s + 1$.
- If $k = dp - 1$, then $h = d - 1$ and $p = k - s + 1$.

To prove assertions (1) and (2) it is thus enough to show that

$$\sum_{i=1}^{r} \gamma_i = k.$$

To prove this we write $k = ar + b$ where $a \in \mathbb{Z}$ and $0 \le b < r$. Then, for $0 \le j \le r-1$, we have

$$\left\lfloor \frac{b+j}{r} \right\rfloor = \begin{cases} 0, & \text{if } j < r-b; \\ 1, & \text{else.} \end{cases}$$

Thus we obtain

$$\begin{aligned} \sum_{i=1}^{r} \gamma_i = \sum_{i=1}^{r} \left\lfloor \frac{k+i-1}{r} \right\rfloor = \sum_{j=0}^{r-1} \left\lfloor \frac{k+j}{r} \right\rfloor = \\ = ar + \sum_{j=0}^{r-1} \left\lfloor \frac{b+j}{r} \right\rfloor = ar + b = \\ = k. \end{aligned}$$

It remains to prove assertion (3), so let us assume $k = dp + 1$. Now it is convenient to fix h and let i vary. For $h \ge 2$, we have $|P_{s,h}| > k$, which implies that we can recover i from the set $I(i,h)$, which means that we can count as above and obtain $\sum_{i=1}^{r}(\gamma_i - 1)$ sets. On the other hand, we also get one additional set $I(i,1) = P_{s,1}$ when $h = 1$. The cardinality of B_s is thus

$$|B_s| = 1 + \sum_{i=1}^{r}(\gamma_i - 1) = 1 + k - r = s = k - p$$

as we claimed.

□

Observe that if $h = d$, the sets $I(i,h)$ are actually intervals of P_s. In particular, $\mathcal{L}_s$ contains all the intervals of length k of P_s.

Proposition 4.5.

- *If* $k \equiv -1 \pmod d$, *then* $|\mathcal{L}_{k-p+1}| = k+1$.
- *If* $k \equiv 0 \pmod d$, *then* $|\mathcal{L}_{k-p+1}| = 1$.
- *If* $k \equiv 1 \pmod d$, *then* $\mathcal{L}_{k-p+1} = \varnothing$.

Proof. For $s = k-p+1$, the set P_s has cardinality dp. If $k = dp-1$, then P_s has $k+1$ distinct k-element subsets, and they are all in $\mathcal{L}_s$ since they are intervals of P_s. If $k = dp$, then P_s itself is its only k-element subset, and it is in $\mathcal{L}_s$ since it is an interval of P_s. Finally, if $k = dp+1$, then P_s has no k-element subsets. □

Proposition 4.6. *We have that* $|\mathbb{I}| = k(dk-k)+1$.

Proof. There are three cases to consider, depending on the congruence class of k modulo d. In each case, we will combine the results of Proposition 4.4 and Proposition 4.5.

If $k \equiv -1 \pmod d$, then

$$\begin{aligned} |\mathbb{I}| &= kd(k-p) + k + 1 = (dp-1)d(dp-1-p) = \\ &= d^3p^2 - 2d^2p - d^2p^2 + d + dp + dp = \\ &= (dp-1)^2(d-1) + 1 = k(dk-k) + 1. \end{aligned}$$

If $k \equiv 0 \pmod d$, then

$$|\mathbb{I}| = kd(k-p) + 1 = d^2p(dp-p) + 1 = k(dk-k) + 1.$$

If $k \equiv 1 \pmod d$, then

$$\begin{aligned} |\mathbb{I}| &= kd(k-p-1) + d(k-p) = kd(k-p) - dp = \\ &= (dp+1)d(dp-p+1) - dp = d^3p^2 + 2d^2p - d^2p^2 - 2dp + d = \\ &= (dp+1)^2(d-1) + 1 = k(dk-k) + 1. \end{aligned}$$

□

Proposition 4.6 shows that $\mathbb{I}$ is a collection of k-element subsets of $[n]$ which has the cardinality of a maximal noncrossing collection, and recall that by construction $\mathbb{I}$ is symmetric. Thus it remains to prove that the sets in $\mathbb{I}$ are pairwise noncrossing.

We start with an immediate consequence of the discussion we already used to count the elements of $\mathcal{L}_s$:

Lemma 4.7. *Assume that* $I = I(i_1, h_1) \overset{n}{+} kx_1 = I(i_2, h_2) \overset{n}{+} kx_2 \in \mathcal{L}_s$*, with* h_1 *and* h_2 *chosen to be minimal. Assume moreover that* $I \neq P_s$*. Then* $(i_1, h_1) = (i_2, h_2)$ *and* $x_1 \equiv x_2 \pmod{n}$*.*

Proof. First, we have that $h_1 = h_2 = |I \cap \overline{a_s}|$. The set $A = I \cap \overline{a_s}$ is an interval of $\overline{a_s}$.

If $A = \overline{a_s}$, then I is a proper interval of P_s, since it is not equal to P_s by assumption. Then $i_1 = i_2$ is the minimal element of this interval. There is moreover a unique (modulo n) integer x such that $i_1 \in [S^{x-1}_{\overline{a_s}}(a_s), S^{x}_{\overline{a_s}}(a_s)]$, and this has to coincide with both x_1 and x_2.

If $A \neq \overline{a_s}$, then A has a minimal element which is equal to both $S^{x_1}_{\overline{a_s}}(a_s)$ and $S^{x_2}_{\overline{a_s}}(a_s)$, so we obtain that $x_1 \equiv x_2 \pmod{n}$. Now i_1 and i_2 are both equal to the minimal element of $[S^{x_1-1}_{\overline{a_s}}(a_s), S^{x_1}_{\overline{a_s}}(a_s)] \cap I$, so they coincide. □

In view of Lemma 4.7, we will often be able to reduce to considering only the case $I = I(i, h) \in B_s$. In the rest of this section, we will always assume that the parameter h is chosen to be minimal.

Lemma 4.8. *Let* $I \in \mathcal{L}_s$*. Let* $a, c \in I$*, and let* $b, d \in P_s \setminus I$ *such that* $a <_\circ b <_\circ c <_\circ d$*. Then* $b \in \overline{a_s}$ *or* $d \in \overline{a_s}$*.*

Proof. By Lemma 4.7, we can assume that $I = I(i, h) \in B_s$. Thus I is an interval of $P_{s,h}$, so by Lemma 3.3 we deduce that b or d has to lie in $P_s \setminus P_{s,h} \subseteq \overline{a_s}$. □

The following two propositions show that the elements of $\mathbb{I}$ are noncrossing.

Proposition 4.9. *If* $I \in \mathcal{L}_s$ *and* $J \in \mathcal{L}_t$ *for* $s \neq t$*, then* I *and* J *are noncrossing.*

Proof. By symmetry, assume $s < t$. Then $J \subseteq [n] \setminus \bigcup_{i=1}^{s} \overline{a_i}$. Assume to reach a contradiction that I and J are crossing. Then there are $a, c \in I \setminus J$ and $b, d \in J \setminus I$ with $a <_\circ b <_\circ c <_\circ d$. By Lemma 4.8, it follows that b or d are in $\overline{a_s}$, which is a contradiction since $b, d \in J$. □

Proposition 4.10. *If* $I, J \in \mathcal{L}_s$*, then* I *and* J *are noncrossing.*

Proof. Assume that I, J are crossing, that is, there exist $a <_\circ b <_\circ c <_\circ d \in P_s$ with $a, c \in I \setminus J$ and $b, d \in J \setminus I$. By applying Lemma 4.8 to first I and then J, we can without loss of generality assume $a, b \in \overline{a_s}$. By Lemma 4.7, we can assume that $J = I(j, h) \in B_s$. Observe that $I \neq J$ by assumption, so $|P_s| > k$ and j is uniquely determined. Let us consider the linear order $<_j$ with minimal element j on P_s. If $d <_j b$, then by construction the interval $[d, b]_{P_s}$ is contained in J. However, we have that $a \in [d, b]_{P_s} \setminus J$, therefore this cannot happen and we must then have $b <_j d$. In particular $c \in [b, d]_{P_s} \subset J \cup \overline{a_s}$, so we conclude that $c \in \overline{a_s}$.

We thus have that $a, b, c \in \overline{a_s}$, with $a, c \in I$. By construction, since I in $\mathcal{L}_s$, we must then have $b \in I$, a contradiction. □

Theorem 4.11. *If* (k, dk) *satisfies Condition* 1.5*, then the collection* $\mathbb{I}$ *constructed above is a symmetric maximal* (k, dk)*-noncrossing collection.*

Proof. By construction, $\mathbb{I}$ is a collection of k-element subsets of $[dk]$. It is symmetric since $\mathcal{L}_s = \mathcal{L}_s \overset{dk}{+} k$ for every s. If $I, J \in \mathbb{I}$, then I and J are noncrossing by Proposition 4.9 and Proposition 4.10. Finally, Proposition 4.6 and Theorem 1.8 imply that $\mathbb{I}$ is a maximal noncrossing collection. □

5. Construction: the general case

Now assume that (k, n) are integers which satisfy Condition 1.5. Set $g = \mathrm{GCD}(k, n)$ and $d = n/g$. Since $\mathrm{GCD}(k, dk) = k$, Condition 1.5 is satisfied for (k, dk) . We will first construct a symmetric maximal (k, dk)-noncrossing collection, then pick a suitable subcollection which will be in bijection with a maximal (k, n)-noncrossing collection.

Choose any linear order on the classes $\bar{1}, \ldots, \bar{g}$ modulo n. Complete it to a linear order on all classes $\bar{1}, \ldots, \bar{k}$ modulo n such that

$$\min\left\{\bar{1}, \ldots, \bar{g}\right\} > \max\left\{\overline{g+1}, \ldots, \bar{k}\right\}.$$

Construct a symmetric maximal (k, dk)-noncrossing collection $\mathbb{I}$ as in Section 4 with this linear order as datum. Define

$$\mathbb{J} = \mathbb{I} \setminus \bigcup_{s=1}^{k-g} \mathcal{L}_s.$$

Thus $\mathbb{J}$ is a collection of k-element subsets of $A = \bigcup_{s=1}^{g} \overline{a_s}$. Notice that A has a cyclic order, induced by that on $[dk]$. For an example of this construction see Section 6.

Observe that, for any $s \in [g]$, the cardinality of $\overline{a_s}$ is the order of k in $\mathbb{Z}/n\mathbb{Z}$, which is $n/g = d$. It follows that $|A| = gd = n$.

Proposition 5.1. *The collection $\mathbb{J}$ is a maximal noncrossing collection of k-element subsets of A which satisfies $\mathbb{J} = \mathbb{J} \overset{dk}{+} k$.*

Proof. The collection $\mathbb{J}$ is by construction a collection of noncrossing subsets of A, since the cyclic order on A is induced by that on $[dk]$. For every s we have that $\mathcal{L}_s = \mathcal{L}_s \overset{dk}{+} k$ by construction, hence $\mathbb{J} = \mathbb{J} \overset{dk}{+} k$ since $\mathbb{J}$ is a union of various $\mathcal{L}_s$. We can compute

$$|\mathbb{J}| = |\mathbb{I}| - \sum_{s=1}^{k-g} \mathcal{L}_s = k(dk - k) + 1 - (k - g)dk = dkg - k^2 + 1 = k(n - k) + 1.$$

Since $|A| = n$, we conclude using Theorem 1.8 that $\mathbb{J}$ is maximal among all collections of noncrossing k-element subsets of A. □

Now define a function $F : [n] \to A$ by

$$F(a + gx) = a + kx$$

for $a \in [g]$ and $x = 0, \ldots, d - 1$. This is well defined and injective by the division algorithm on $[n]$ and $[dk]$ respectively. Since $|A| = n$, we conclude that F is bijective.

Lemma 5.2. *We have $F \circ S_{[n]} = S_A \circ F$.*

Proof. The function F is increasing, so it preserves the linear orders on $[n]$ and A with minimum elements equal to 1. Since $F(1) = 1$, we conclude that $S_A(F(x)) = F(x + 1)$ for all $1 \leq x \leq n - 1$. But $S_A(g + k(d - 1)) = 1 = F(n + 1)$, which proves the claim. □

We extend F to subsets of $[n]$ and still call it F. It is a bijection between subsets of $[n]$ and subsets of A. Call $\mathbb{I}' = F^{-1}(\mathbb{J})$.

Proposition 5.3. *We have $\mathbb{I}' = \mathbb{I}' \overset{n}{+} k$.*

Proof. Pick $I \in \mathbb{I}'$. Then by Proposition 5.1 we have $F(I) \overset{dk}{+} k \in \mathbb{J}$. Then

$$F^{-1}(F(I) \overset{dk}{+} k) = I \overset{n}{+} g \in \mathbb{I}'$$

so that $\mathbb{I}' = \mathbb{I}' \overset{n}{+} g$. Since $k = g \cdot \frac{k}{g}$ and $\frac{k}{g}$ is an integer, we are done. □

Theorem 5.4. *If (k, n) satisfies Condition* 1.5*, then the collection $\mathbb{I}'$ constructed above is a symmetric maximal (k, n)-noncrossing collection.*

Proof. By construction, $\mathbb{I}'$ is a collection of k-element subsets of $[n]$. It is symmetric by Proposition 5.3. It is a maximal (k, n)-noncrossing collection by Proposition 5.1 and Lemma 5.2. □

We can now prove Theorem 1.6.

Proof of Theorem 1.6. By Proposition 2.1, Condition 1.5 is necessary. By Theorem 5.4, the collection constructed in Section 5 is a symmetric maximal (k, n)-noncrossing collection, so Condition 1.5 is also sufficient. □

Remark 5.5. There exist symmetric maximal noncrossing collections which do not come from our construction. Indeed, one can produce $\mathrm{GCD}(k, n)!/(p - 1)!$ different collections by varying the total order on $\{\overline{a_1}, \ldots, \overline{a_k}\}$ to define the various $\mathcal{L}_s$. However, a computer search produces the lower bounds for the number of distinct symmetric maximal (k, n)-noncrossing collections shown in the following table.

$k \setminus n$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
2	2		2												
3			6			24			24						
4					110		6		> 894				> 1900		
5							> 2000					> 4800			
6									> 4900		18	> 840			> 5000
7											> 5000				
8													> 5000		54

The numbers without a $>$ sign are exact. It is easy to check by hand that, for instance, our construction produces only 2 of the 6 symmetric maximal $(4, 10)$-noncrossing collections.

6. Example

In this section we illustrate our construction with an example. Suppose we want to construct a symmetric maximal $(4, 10)$-noncrossing collection. Since $n = 10$ is not a multiple of $k = 4$, we need to use the procedure described in Section 5. Thus we will first construct a symmetric maximal $(4, 20)$-noncrossing collection, where $g = 2$, $d = 5$, and $20 = kd$.

We pick the following order on congruence classes modulo 20:

$$\bar{3} < \bar{4} < \bar{2} < \bar{1}.$$

Then we get the set

$$B_1 = \{\{20, 1, 2, 3\}, \{1, 2, 3, 4\}, \{2, 3, 4, 5\}, \{3, 4, 5, 6\}\}.$$

Notice that all these sets contain the element 3. The set $\mathcal{L}_1 = B_1 \overset{20}{+} 4\mathbb{Z}$ consists of the 20 intervals of $[20]$.

The next step is removing the orbit $\bar{3} = \{3, 7, 11, 15, 19\}$ and constructing

$$B_2 = \{\{1, 2, 4, 5\}, \{2, 4, 5, 6\}, \{4, 5, 6, 8\}, \{4, 5, 6, 9\}\}.$$

Again, we define $\mathcal{L}_2 = B_2 \overset{20}{+} 4\mathbb{Z}$.

Next we remove the orbit of 4 and construct

$$B_3 = \{\{1, 2, 5, 6\}, \{1, 2, 5, 9\}, \{2, 5, 6, 9\}, \{2, 5, 9, 13\}\}$$

and $\mathcal{L}_3 = B_3 \overset{20}{+} 4\mathbb{Z}$.

Finally, when the only orbit left is $\bar{1} = \{1, 5, 9, 13, 17\}$ we get

$$B_4 = \{\{1, 5, 9, 13\}\}$$

so that $\mathcal{L}_4$ consists of the 5 different shifts of $\{1, 5, 9, 13\}$.

By Theorem 4.11, the collection $\mathbb{I} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3 \cup \mathcal{L}_4$ is a symmetric maximal $(4, 20)$-noncrossing collection. Now we restrict our attention to $\mathbb{J} = \mathcal{L}_3 \cup \mathcal{L}_4$, which is a maximal noncrossing collection in the cyclically ordered set $\bar{1} \cup \bar{2} \subseteq [20]$. Observe that $\mathbb{J}$ has 25 elements, which is indeed $4(10 - 4) + 1$ (cf. Theorem 1.8). It remains to rename the elements of $\mathbb{J}$ to obtain a symmetric maximal noncrossing collection in $[10]$. The function F defined in Section 5 maps $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$ to $(1, 2, 5, 6, 9, 10, 13, 14, 17, 18)$, so the collection $\mathbb{I}' = F^{-1}(\mathcal{L}_3 \cup \mathcal{L}_4)$ is

$$\{\{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{2, 3, 4, 5\}, \{2, 3, 5, 7\}, \{1, 3, 5, 7\}\} \overset{10}{+} 2\mathbb{Z}.$$

As per Theorem 5.4, this is a maximal $(4, 10)$-noncrossing collection invariant under adding 2 modulo 10, and thus invariant under adding 4 modulo 10 as we wanted.

7. Algebraic consequences

In this section we illustrate some representation-theoretic consequences of Theorem 1.6.

There is a way of generating a quiver Q (i.e., a directed graph) embedded in $\mathbb{R}^2$, starting from a maximal (k, n)-noncrossing collection $\mathbb{I}$. As a graph, Q is nothing but the 1-skeleton of the CW-complex $\Sigma(\mathbb{I})$ of Section 2. The edges are oriented such that the face to the right is the one whose label has size $k - 1$. As was observed in Section 2, this quiver is invariant under rotation by $\frac{2\pi k}{n}$ if and only if $\mathbb{I}$ is symmetric.

One can define a potential W on Q by taking the sum of all clockwise faces minus the sum of all anticlockwise faces. Thus (Q, W) becomes a quiver with potential in the sense of [DWZ08]. By taking the boundary vertices (those corresponding to the intervals of $[n]$) as frozen, one can then define the frozen Jacobian algebra A [BIRS11, Definition 1.1].

The main result of [BKM16] is that $A \cong \operatorname{End}_B(T)$, where $B = B(k,n)$ is an infinite-dimensional algebra introduced in [JKS16] and T is a cluster tilting object in the category $\mathrm{CM}(B)$. In fact, one can associate a rank one Cohen-Macaulay B-module L_I to each $I \subseteq [n]$ of size k (see [JKS16, §5]), and take $T = \bigoplus_{i\in\mathbb{I}} L_I$. It is indeed proved in [JKS16, Proposition 5.6] that I and J are noncrossing if and only if $\operatorname{Ext}^1(L_I, L_J) = 0 = \operatorname{Ext}^1(L_J, L_I)$.

One can also look at the stable category $\underline{\mathrm{CM}}(B)$ of $\mathrm{CM}(B)$, which is triangulated and in fact 2-Calabi-Yau [JKS16, Corollary 4.6] and [GLS08, Proposition 3.4]. The triangulated structure of this category plays a crucial role in the motivations behind this article: we have that $L_I[-2] \cong L_{I\overset{n}{+}k}$ in $\underline{\mathrm{CM}}(B)$ [BB17, Proposition 2.7],[JKS16, §7]. Thus saying that $\mathbb{I} = \mathbb{I} \overset{n}{+} k$ is equivalent to saying that $T \cong T[2]$.

Remark 7.1. In [BKM16] and [Pas17], the focus is on Postnikov diagrams (or alternating strand diagrams). By [OPS15, Theorem 11.1], there is a bijection between Postnikov diagrams and maximal noncrossing collections, so the two concepts are interchangeable. In this article we focus on collections since constructing a maximal noncrossing collection explicitly is much easier than constructing a Postnikov diagram.

If we denote by $e \in A$ the idempotent corresponding to the boundary vertices of Q, we have that $A/AeA \cong \operatorname{End}_{\underline{\mathrm{CM}}(B)}(T)$ by [Pas17, Lemma 6.5]. One can prove (see [Pas17, Proposition 4.2]) that A/AeA is self-injective if and only if $T \cong T[2]$, which as we saw holds if and only if $\mathbb{I}$ is symmetric. The algebra $\Lambda = \Lambda(\mathbb{I}) = A/AeA$ is the Jacobian algebra of the quiver with potential obtained rom (Q,W) by removing the boundary vertices. Observe that these correspond to the intervals of $[n]$, which are part of all maximal noncrossing collections. Hence all the information carried by $\mathbb{I}$ is preserved by looking only at the sets in $\mathbb{I}$ which are not intervals.

The original motivation of this work was to find examples of self-injective Jacobian algebras. This interest stems in turn from higher dimensional Auslander-Reiten theory, in which these algebras play a role analogous to that of preprojective algebras of Dynkin quivers (see [HI11]). One consequence of Theorem 1.6 is that there exist many such algebras (which is a priori unclear, cf. [HI11, Question 10.1]).

Corollary 7.2. *Let (k,n) be a pair satisfying Condition* 1.5*, and let $B = B(k,n)$ be the algebra defined in* [JKS16, §3]*. Then:*

1. *there exists a cluster tilting module $T \cong T[2] \in \underline{\mathrm{CM}}(B)$ whose indecomposable summands are rank one modules;*
2. *the algebra $\Lambda = \operatorname{End}_{\underline{\mathrm{CM}}(B)}(T)$ is a self-injective Jacobian algebra;*
3. *a Nakayama automorphism of Λ is induced by $I \mapsto I - k$.*

Proof. By Theorem 1.6, there exists a maximal (k,n)-noncrossing collection $\mathbb{I} = \mathbb{I} \overset{n}{+} k$. Take $T = \bigoplus_{i\in\mathbb{I}} L_I$, then the statements follow from [Pas17, Theorem 7.2]. □

Another interesting consequence of Theorem 1.6 comes from looking at the order of the Nakayama automorphism given in Corollary 7.2(3). Suppose we fix d and we want to construct examples of self-injective Jacobian algebras with Nakayama automorphism of order d. One possibility that works for every $d \geq 3$ is to take a cluster tilted algebra [BIRS11], [Kel11]. A result by Ringel [Rin08] classified the self-injective ones, and it turns out that for a fixed d there at most two such algebras with Nakayama automorphism of order d. All the other examples presented in [HI11] have Nakayama automorphism of order 2 or 3 (for these, infinite families are given). Sporadic examples of order 4 and 5 have later been found by Herschend and Lakani independently. An example of a self-injective Jacobian algebra with Nakayama automorphism of order $2x+1$ for any $x \in \mathbb{Z}_{>0}$ is constructed in [Pas17] using Postnikov diagrams. As a corollary of our construction, we get:

Corollary 7.3. *Let $d \in \mathbb{Z}_{>1}$. There exist infinitely many self-injective Jacobian algebras with a Nakayama automorphism of order d.*

Proof. Choose $d \in \mathbb{Z}_{>1}$. There are infinitely many choices of (k,n) satisfying Condition 1.5 such that $d = n/\operatorname{GCD}(k,n)$. Indeed, take for instance

$$(k,n) \in \left\{(d,d^2),(2d,2d^2),(3d,3d^2),\dots\right\}.$$

By Corollary 7.2, for each of these pairs there exists a self-injective Jacobian algebra with Nakayama automorphism of order the order of $a \mapsto a - k$ on $\mathbb{Z}/n\mathbb{Z}$, that is, d. They are pairwise non-isomorphic, since their quivers have different numbers of vertices. □

Remark 7.4. One can also choose the families

$$(k,n) \in \{(d\pm 1, d(d\pm 1)),(2(d\pm 1),2d(d\pm 1)),(3(d\pm 1),3d(d\pm 1)),\dots\}$$

in the proof of Corollary 7.3, to get other infinite families of self-injective Jacobian algebras with Nakayama automorphism of order d. In the latter families, the Nakayama permutation acts freely on the vertices of the quiver, while in the family used in the proof there is a fixed vertex.

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Skew group algebras of Jacobian algebras

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ABSTRACT

For a quiver with potential (Q, W) with an action of a finite cyclic group G, we study the skew group algebra ΛG of the Jacobian algebra $\Lambda = \mathcal{P}(Q, W)$. By a result of Reiten and Riedtmann, the quiver Q_G of a basic algebra $\eta(\Lambda G)\eta$ Morita equivalent to ΛG is known. Under some assumptions on the action of G, we explicitly construct a potential W_G on Q_G such that $\eta(\Lambda G)\eta \cong \mathcal{P}(Q_G, W_G)$. The original quiver with potential can then be recovered by the skew group algebra construction with a natural action of the dual group of G. If Λ is self-injective, then ΛG is as well, and we investigate this case. Motivated by Herschend and Iyama's characterisation of 2-representation finite algebras, we study how cuts on (Q, W) behave with respect to our construction.
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Contents

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1. Introduction

The aim of this article is to study the skew group algebra of a Jacobian algebra coming from a quiver with potential. Skew group algebras were first studied from the point of view of representation theory in [19]. If Λ is a finite-dimensional algebra over a field k with a finite group G acting on Λ by automorphisms, then the skew group algebra ΛG shares many representation-theoretic properties with Λ, often incarnated in properties of functors between $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Lambda G$. If Λ is the quotient of a path algebra by an admissible ideal, then Reiten and Riedtmann describe the quiver Q_G of (a basic version of) ΛG. This description is complete if G is cyclic, and Demonet extended it to a complete description for arbitrary finite groups if Λ is hereditary [4]. However, describing the relations on this quiver is difficult in general.

Something can be said for Jacobian algebras of quivers with potential (QPs). These were introduced in [5], and have since found applications in cluster theory via the Amiot cluster category [1]. An action of G on a QP (Q, W) induces an action on the corresponding Ginzburg dg algebra defined in [6]. In [14], it is shown that the skew group dg algebra is Morita equivalent to the Ginzburg dg algebra associated to another QP. The quiver is Q_G, and the potential is the image of W under a natural map. Moreover, in [14, §4.5] the potential is expressed as a linear combination of cycles of Q_G in some examples.

In this article we eschew the dg setting and focus on Jacobian algebras of QPs. The Jacobian algebra $\mathcal{P}(Q, W)$ is the 0-th homology of the Ginzburg dg algebra, and can be identified with the algebra obtained by imposing on Q the relations coming from all cyclic derivatives of W. In particular, Le Meur's result implies that $\mathcal{P}(Q,W)G$ is Morita equivalent to the Jacobian algebra of a QP. Under some assumptions on the action, we explicitly construct a potential W_G on Q_G such that we have:

Theorem *(Theorem 3.20). Let (Q,W) be a QP, and let $\Lambda = \mathcal{P}(Q,W)$. Let G be a finite cyclic group acting on (Q,W) as per the assumptions (A1)–(A7) of §3.1. Let Q_G be the quiver constructed in §3.2, W_G the potential on Q_G defined in §3.3, and $\eta \in \Lambda G$ the idempotent defined in §3.1. Then*

$$\mathcal{P}(Q_G, W_G) \cong \eta(\mathcal{P}(Q,W)G)\eta.$$

As observed in [19, §5], there is a natural action of the dual group $\hat{G}$ on ΛG. In our setting, this action restricts to the basic algebra $\eta(\Lambda G)\eta$. Reiten and Riedtmann prove that $(\Lambda G)\hat{G}$ is Morita equivalent to Λ if G is abelian, so it is natural to ask whether one gets back the original QP by applying this second skew group algebra construction. To do so, one needs to find assumptions which guarantee that W_G is fixed by $\hat{G}$ as an element of $kQ_G \cong \eta((kQ)G)\eta$, and which are preserved under taking skew group algebras. If $G = \mathbb{Z}/2\mathbb{Z}$, it was shown in [2] that indeed we get (Q,W) back (and in fact the Ginzburg dg algebra of (Q,W)). We extend Amiot and Plamondon's result to our setting (assumptions (A1)–(A7) of §3.1), via a direct check using our formula for W_G:

Theorem *(Proposition 5.3 and Corollary 5.4). There is an isomorphism of quivers $\phi : (Q_G)_{\hat{G}} \cong Q$ such that, if we extend it to an isomorphism between the corresponding path algebras, we have $\phi((W_G)_{\hat{G}}) = W$. This induces an algebra isomorphism*

$$\theta\left(\left(\eta\left(\Lambda G\right)\eta\right)\hat{G}\right)\theta \cong \Lambda,$$

where $\Lambda = \mathcal{P}(Q,W)$ and θ is the idempotent defined in Section 5.

A simple example of the above construction which is good to have in mind is illustrated in Example 8.1, and specifically in the quivers of Fig. 4 and Fig. 5. Here we take Q to be the QP of the 3-preprojective algebra of type A_4, so the potential is given by the sum of all 3-cycles with alternating signs. The group $G = \mathbb{Z}/3\mathbb{Z}$ acts by rotations in the plane, and the quiver Q_G is given in Fig. 5. Here the action of $\hat{G}$ permutes the vertices 4^i and multiplies the arrow $\tilde{\delta}$ by a third root of unity, and one can check that by performing the same construction on Q_G one gets Q back.

We look with special interest at the case where $\Lambda = \mathcal{P}(Q,W)$ is self-injective. This is because of the relationship between self-injective QPs and higher (in this case 2-) Auslander–Reiten theory. Namely, self-injective Jacobian algebras are precisely the 3-preprojective algebras of 2-representation finite algebras (see [8]). Any such 2-representation finite algebra can be constructed from (Q,W) together with the combinatorial datum of a so-called cut, as a truncated Jacobian algebra. In [19] it is proved that the skew group algebra construction preserves self-injectivity. We show that it also preserves the property of being Frobenius, and compute a Nakayama automorphism of ΛG if the bilinear form on Λ is G-equivariant. As a consequence, we prove that if Λ is the Jacobian algebra of a planar self-injective QP and we take G generated by a Nakayama automorphism, then ΛG is symmetric. We show that G-invariant cuts on (Q, W) induce cuts on (Q_G, W_G), and the corresponding truncated Jacobian algebras are obtained from each other by a skew group algebra construction. Thus we have that, under some hypotheses, 2-representation finiteness is preserved under taking skew group algebras (note that an analogous result was obtained, using different methods, for some d-representation infinite algebras in [7]). Moreover we give some sufficient conditions on (Q, W) which imply that all the truncated Jacobian algebras of (Q_G, W_G) are derived equivalent. It was recently shown in [14], by different methods, that in fact the property of being d-representation (in)finite is always preserved under taking skew group algebras. An example where the 2-representation finite algebra is constructed from tensor product of Dynkin quivers is illustrated in Example 8.6. We also look at a case where Λ is not self-injective in Example 8.7. Here we realise an Auslander algebra as a truncated Jacobian algebra, thus checking directly a special case of [19, Theorem 1.3(c)(iv)].

There is a natural class of QPs with a group action satisfying our assumptions, namely rotation-invariant planar QPs. Planar QPs were introduced in [8] as they behave particularly nicely when they have self-injective Jacobian algebras. It turns out that in all known examples of self-injective planar QPs a Nakayama automorphism acts by a rotation, hence they fit nicely in our setting. Recently it has been shown that Postnikov diagrams have connections with planar self-injective QPs: in [16] it is proved that the QP coming from an (a, n)-Postnikov diagram on a disk (as in [3]) is self-injective if and only if the diagram is rotation invariant. Thus, our construction produces many examples of symmetric Jacobian algebras, one for every such Postnikov diagram. An example is given as Example 8.3.

The structure of this article is as follows. In Section 2 we recall definitions and some facts about quivers with potential and skew group algebras. Moreover, we prove that skew group algebras of Frobenius algebras are again Frobenius. In Section 3 we explain our setup and assumptions, fix notation and present our main result. Section 4 is devoted to proving Theorem 3.20. In Section 5 we look at the $\hat{G}$-action on $\mathcal{P}(Q_G, W_G)$, and prove that we get back to (Q, W) by taking the skew group algebra with respect to this action. In Section 6 we apply our results to planar rotation-invariant QPs. In Section 7 we consider how cuts behave with respect to taking skew group algebras, and the consequences for truncated Jacobian algebras. Section 8 consists of some examples which illustrate our construction.

2. Preliminaries

2.1. Conventions

We denote by k a fixed field. Algebras are assumed to be associative unital finite dimensional k-algebras. We denote by $D = \mathrm{Hom}_k(-, k)$ the k-dual, in both directions. Quivers are understood to be finite and connected. For a quiver Q, we denote by Q_0 its set of vertices and by Q_1 its set of arrows. We compose quiver arrows from right to left, as functions. For an arrow α, we denote by $\mathfrak{s}(\alpha)$ and $\mathfrak{t}(\alpha)$ its start and target respectively. We compose quiver arrows from right to left, as functions. For an arrow α, we denote by $\mathfrak{s}(\alpha)$ and $\mathfrak{t}(\alpha)$ its start and target respectively. If p is a path in a quiver and α is an arrow, we use the notation $\alpha \in p$ to indicate that α appears as one of the arrows in p. A relation of a quiver is a linear combination of paths with the same start and end.

Let Λ be an algebra and $\varphi \colon \Lambda \to \Lambda$ be an algebra endomorphism. For a right Λ-module M, we define M_φ to be the right Λ-module which is equal to M as a vector space but whose action is given by $m \cdot \lambda = m\varphi(\lambda)$, for all $m \in M$ and $\lambda \in \Lambda$. For a finite group G, we denote by kG the corresponding group algebra. If X is a subset of a ring A, we denote by $\langle X \rangle$ the two-sided ideal of A generated by X.

2.2. Index of terminology

Since the statements, constructions and proofs in this article are quite technical and notation-heavy, we collect here the main terminology we use. The definitions given here are not meant to be complete, but we refer to the position in the text where they are explained properly.

Symbol	Description	Reference
$a(c)$	The coefficient of the cycle c in W.	Notation 3.1
$*$	The "forgetful" action of G on Q.	Notation 3.2
$b(\alpha)$	For an arrow α between fixed vertices, we define $b(\alpha)$ by $g(\alpha) = \zeta^{b(\alpha)}\alpha$.	Notation 3.4
Types (i)–(iv)	The types of cycles appearing in W, as per assumption (A7).	Notation 3.6
e_μ	A choice of idempotents of kG.	Notation 3.9
$\mathcal{E}, \mathcal{E}', \mathcal{E}''$	Chosen subsets of Q_0.	Notation 3.10
η	The Morita idempotent we choose for ΛG.	Notation 3.11
Types (1)–(4)	Types of arrows. Every arrow in Q is in the G-orbit of an arrow of one of these types.	Notation 3.13
$\tilde{\alpha}, \tilde{\alpha}^\mu$	The arrows of Q_G we define.	Notation 3.13
$t(\alpha)$	For α of type (1), $\mathfrak{s}(\alpha) \in g^{t(\alpha)}(\mathcal{E})$. Otherwise, $t(\alpha) = 0$.	Notation 3.15
$\hat{c}$	A chosen cycle in the G-orbit of a cycle c.	Notation 3.17
$\tilde{c}, \tilde{c}^\mu$	Cycles in Q_G we define.	Notation 3.17
$p(c), q(c)$	Integers associated to cycles of type (ii) and (iii) respectively.	Notation 3.17
W_G	The potential we define on Q_G.	Notation 3.18

2.3. Skew group algebras

Let G be a finite group acting on an algebra Λ by automorphisms.

Definition 2.1. The *skew group algebra* ΛG is the algebra defined by:

- its underlying vector space is $\Lambda \otimes_k kG$;
- multiplication is given by

$$(\lambda \otimes g)(\mu \otimes h) = \lambda g(\mu) \otimes gh$$

  for $\lambda, \mu \in \Lambda$ and $g, h \in G$, extended by linearity and distributivity.

There is a natural algebra monomorphism $\Lambda \to \Lambda G$ given by $\lambda \mapsto \lambda \otimes 1$. Notice that the algebra ΛG is not basic in general.

2.4. Quivers with potential

We follow [8] in our presentation. Let Q be a quiver. Denote by $\widehat{kQ}$ the completion of kQ with respect to the $\langle Q_1 \rangle$-adic topology. Define

$$\mathrm{com}_Q = \overline{\left[\widehat{kQ}, \widehat{kQ}\right]} \subseteq \widehat{kQ},$$

where $\overline{}$ denotes closure. Thus $\widehat{kQ}/\,\mathrm{com}_Q$ has a topological basis consisting of cycles in Q. In particular there is a unique continuous linear map

$$\sigma : \widehat{kQ}/\,\mathrm{com}_Q \to \widehat{kQ}$$

induced by

$$\alpha_1 \cdots \alpha_n \mapsto \sum_{m=1}^{n} \alpha_m \cdots \alpha_n \alpha_1 \cdots \alpha_{m-1}.$$

For each $\alpha \in Q_1$ define $d_\alpha : \langle Q_1 \rangle \to \widehat{kQ}$ to be the continuous linear map given by $d_\alpha(\alpha p) = p$ and $d_\alpha(q) = 0$ if q does not end with α. Define the *cyclic derivative* with respect to an arrow α to be $\partial_\alpha = d_\alpha \circ \sigma : \langle Q_1 \rangle/\,\mathrm{com}_Q \to \widehat{kQ}$. It will be convenient to take derivatives with respect to multiples of arrows. For $\lambda \in k^*$, define $\partial_{\lambda\alpha}(c) = \lambda^{-1}\partial_\alpha(c)$. A *potential* is an element $W \in \langle Q_1 \rangle^3/(\langle Q_1 \rangle^3 \cap \mathrm{com}_Q)$, i.e., a (possibly infinite) linear combination of cycles of length at least 3. A potential is called *finite* if it can be written as a finite linear combination of cycles. By an abuse of notation, if c is a cycle in Q we will denote again by c the corresponding element of $\langle Q_1 \rangle^3/(\langle Q_1 \rangle^3 \cap \mathrm{com}_Q)$ and consider it up to cyclic permutation of its arrows. We call the pair (Q, W) a *quiver with potential (QP)* and define its Jacobian algebra to be

$$\mathcal{P}(Q,W) = \widehat{kQ} \Big/ \overline{\langle \partial_\alpha W \mid \alpha \in Q_1 \rangle}.$$

In our setting, the completion will not play any role, due the following proposition.

Proposition 2.2 *([16, Proposition 2.3]). If W is a finite potential and the ideal $\langle \partial_\alpha W \mid \alpha \in Q_1 \rangle \subseteq kQ$ is admissible, then*

$$\mathcal{P}(Q,W) \cong kQ \Big/ \langle \partial_\alpha W \mid \alpha \in Q_1 \rangle.$$

2.5. Self-injective algebras

We need some facts and notation for Frobenius and self-injective algebras, see for instance [15] or [10]. An algebra Λ is *self-injective* if it is injective as a right Λ-module. It is *Frobenius* if there is a bilinear form $(-,-)$ on Λ which is nondegenerate and multiplicative (i.e., $(a,bc) = (ab,c)$ for all $a,b,c \in \Lambda$). It is *symmetric* if this form can be taken to be symmetric. Frobenius algebras are self-injective, and the converse is true if and only if $\dim \operatorname{Hom}_\Lambda(S,\Lambda) = \dim S$ for all simples S. In particular, self-injective basic algebras are exactly the Frobenius basic algebras.

If Λ is Frobenius, then from the nondegenerate bilinear form we get an isomorphism $f : \Lambda \to D\Lambda$ of vector spaces, given by $f(v) = (-,v)$. Moreover f is an isomorphism of left Λ-modules since

$$f(\lambda v) = (-,\lambda v) = (-\lambda, v) = \lambda(f(v)).$$

Nondegeneracy of the form implies that there exists a unique k-linear map $\varphi : \Lambda \to \Lambda$ satisfying

$$(a,b) = (b,\varphi(a))$$

for all $a,b \in \Lambda$. In fact such a φ is an algebra automorphism, and f becomes a right module isomorphism $f : \Lambda_\varphi \to D\Lambda$. If we choose a different bilinear form and hence a different isomorphism $g : \Lambda \to D\Lambda$ of vector spaces, then $g(a) = f(au)$ for some unit $u \in \Lambda$. Then the corresponding automorphism ψ is given by $\psi(a) = u\varphi(a)u^{-1}$, so φ is unique as an outer automorphism of Λ. The automorphism φ is called a Nakayama automorphism of Λ. In particular, Λ is symmetric if and only if $\varphi = \mathrm{id}_{\mathrm{Out}(\Lambda)}$.

We are interested in studying skew group algebras of Frobenius algebras, and in particular the case where G is generated by a Nakayama automorphism.

Remark 2.3. In [19, Theorem 1.3(c)(iii)] it is proved that skew group algebras of self-injective algebras are always self-injective. In the discussion that follows we show that the property of being Frobenius is also preserved under taking skew group algebras.

Let G be a finite group acting on a Frobenius algebra Λ by automorphisms. The algebra kG is always Frobenius and in fact symmetric. We denote by $(-,-)$ the corresponding symmetric nondegenerate bilinear form on kG as well. This form can be taken to be $(h,l) = \delta_{hl^{-1}}$ for $h, l \in G$, extended bilinearly. Then we can define a bilinear form $\langle -,-\rangle$ on the skew group algebra ΛG by setting

$$\langle \lambda \otimes l, \mu \otimes m\rangle = (\lambda, l(\mu))(l,m)$$

for $\lambda, \mu \in \Lambda$ and $l, m \in G$, extended bilinearly.

Lemma 2.4. *The form $\langle -,-\rangle$ is multiplicative and nondegenerate. In particular, ΛG is Frobenius.*

Proof. We have

$$\begin{aligned}\langle (\lambda \otimes l)(\mu \otimes m), \nu \otimes n\rangle &= \langle \lambda l(\mu) \otimes lm, \nu \otimes n\rangle = \\ &= (\lambda l(\mu), (lm)(\nu))(lm, n) = \\ &= (\lambda, l(\mu) l(m(\nu)))(l, mn) = \\ &= (\lambda, l(\mu m(\nu)))(l, mn) = \\ &= \langle \lambda \otimes l, \mu m(\nu) \otimes mn\rangle = \\ &= \langle \lambda \otimes l, (\mu \otimes m)(\nu \otimes n)\rangle\end{aligned}$$

for all $\lambda, \mu, \nu \in \Lambda$ and $l, m, n \in G$. This proves multiplicativity.

Assume now that there exists $\sum_i \xi_i \otimes z_i \in \Lambda G$ such that $\langle \sum_i \xi_i \otimes z_i, x\rangle = 0$ for all $x \in \Lambda G$. Without loss of generality we can take every z_i to be an element of G. Take $x = \lambda \otimes l$ with $\lambda \in \Lambda$ and $l \in G$. Then

$$0 = \sum_i \langle \xi_i \otimes z_i, \lambda \otimes l\rangle = \sum_i (\xi_i, z_i(\lambda))(z_i, l) = \sum_{z_i = l^{-1}} (\xi_i, l^{-1}(\lambda)) = \left(\sum_{z_i = l^{-1}} \xi_i, l^{-1}(\lambda) \right).$$

Since l acts by an automorphism and $(-,-)$ is nondegenerate, it follows that $\sum_{z_i = l^{-1}} \xi_i = 0$. By iterating this argument for all possible values of l, we get that

$$\sum_i \xi_i \otimes z_i = \sum_{l \in G} \left(\sum_{z_i = l^{-1}} \xi_i \right) \otimes l^{-1} = 0.$$

Assume instead that $\langle x, \sum_i \xi_i \otimes z_i\rangle = 0$ for all $x \in \Lambda G$. Again we suppose that $z_i \in G$ and we take $x = \lambda \otimes l$ with $\lambda \in \Lambda$ and $l \in G$. Then

$$0 = \sum_i \langle \lambda \otimes l, \xi_i \otimes z_i \rangle = \sum_i (\lambda, l(\xi_i))(l, z_i) = \sum_{z_i = l^{-1}} (\lambda, l(\xi_i)) = \left(\lambda, l\left(\sum_{z_i = l^{-1}} \xi_i\right)\right)$$

so that $\sum_{z_i=l^{-1}} \xi_i = 0$ and we can argue as above. This proves nondegeneracy. □

If the bilinear form on Λ is G-equivariant, we can find a Nakayama automorphism of ΛG. Let us choose a Nakayama automorphism φ of Λ.

Proposition 2.5. *If $(g(\lambda), g(\mu)) = (\lambda, \mu)$ for all $g \in G, \lambda, \mu \in \Lambda$, then $\varphi \otimes 1$ is a Nakayama automorphism of ΛG.*

Proof. Let $\lambda, \mu \in \Lambda$ and $l, m \in G$. Then

$$\begin{aligned}\langle \lambda \otimes l, \mu \otimes m \rangle &= \delta_{lm^{-1}}(\lambda, l(\mu)) = \\ &= \delta_{lm^{-1}}(l(\mu), \varphi(\lambda)) = \\ &= \delta_{lm^{-1}}(\mu, l^{-1}\varphi(\lambda)) = \\ &= \delta_{lm^{-1}}(\mu, m\varphi(\lambda)) = \\ &= \langle \mu \otimes m, \varphi(\lambda) \otimes l \rangle. \qquad \square\end{aligned}$$

Corollary 2.6. *If φ generates the image $\mathrm{im}(G) \subseteq \mathrm{Aut}(\Lambda)$, then ΛG is symmetric.*

Proof. Since φ is an element in $\mathrm{im}(G)$, we know that there is an $h \in G$ which acts on Λ as φ. Now let $g \in G$. By assumption, there exists an integer j such that g acts on Λ as φ^j. Then we have

$$(\lambda, \mu) = (\mu, \varphi(\lambda)) = (\varphi(\lambda), \varphi(\mu)) = (\varphi^j(\lambda), \varphi^j(\mu)) = (g(\lambda), g(\mu)),$$

so we can apply Proposition 2.5 and get that

$$\varphi \otimes 1 : \lambda \otimes l \mapsto h(\lambda) \otimes l$$

is a Nakayama automorphism of ΛG. Notice now that $h(\lambda) \otimes l = (1 \otimes h)(\lambda \otimes l)(1 \otimes h)^{-1}$, so that $\varphi \otimes 1$ is the identity as an outer automorphism of ΛG, which means that ΛG is symmetric. □

We include the following lemma, which we will use in Section 6.

Lemma 2.7. *Let Λ be a symmetric algebra, and $e \in \Lambda$ an idempotent. Then $e\Lambda e$ is symmetric.*

Proof. Let $\langle -, - \rangle$ be a symmetric multiplicative nondegenerate bilinear form on Λ. Then the restricted form on $e\Lambda e$ is a symmetric multiplicative bilinear form on $e\Lambda e$. Let now $u \in e\Lambda e$ such that $\langle u, - \rangle_{|e\Lambda e} = 0$. Let $v \in \Lambda$ and observe that

$$\langle u, v\rangle = \langle eue, v\rangle = \langle eu, ev\rangle = \langle ev, eu\rangle = \langle eve, u\rangle = 0$$

so that $u = 0$ since the form is nondegenerate on Λ. $\square$

3. Setup and result

In this section we set up our assumptions, and fix the notation we need to be able to state our results.

Let (Q, W) be a quiver with potential and let Λ be its Jacobian algebra.

Notation 3.1. We write $W = \sum_c a(c)c$.

Recall that we consider cycles up to cyclic permutation. We assume that W is finite and that the cyclic derivatives of W generate an admissible ideal of kQ. In what follows we will freely use integers as indices for convenience, even when they should be seen as elements of $\mathbb{Z}/n\mathbb{Z}$.

3.1. Assumptions

Let G be a cyclic group of order n with generator g, acting on kQ. We make the following assumptions (A1)–(A7).

(A1) The field k contains a primitive n-th root of unity ζ. In particular, $n \neq 0$ in k.
(A2) The action of G permutes the vertices of Q and maps every arrow to a multiple of an arrow.
(A3) Every vertex of Q which is not fixed by G has an orbit of cardinality n.
(A4) We have $GW = W$ in $\widehat{kQ}/\operatorname{com}_Q$.

Since G preserves the potential, we get an induced action of G on Λ.

Notation 3.2. We define a second "forgetful" action $*$ of G on Q by $g * v = g(v)$ for $v \in Q_0$ and $g * \alpha = \beta$ whenever β is an arrow and $g(\alpha)$ is a scalar multiple of β.

Remark 3.3. Let $u, v \in Q_0$ be (not necessarily distinct) vertices fixed by G. The vector space V spanned by arrows from u to v is a kG-module, and since G is abelian it decomposes into 1-dimensional submodules. This means that, up to choosing a different basis for V, we can assume that arrows between fixed vertices are mapped to scalar multiples of themselves.

By this observation, we can without loss of generality make the additional assumption:

(A5) If α is an arrow between two fixed vertices, then $g(\alpha) = \zeta^{b(\alpha)}\alpha$ for an integer $b(\alpha)$.

Notation 3.4. We define $b(\alpha)$ as above, for α any arrow between fixed vertices.

Remark 3.5. Suppose that an arrow α is such that $g(\alpha) = \zeta^i \beta$ for some arrow $\beta \neq \alpha$. Then, by assumptions (A5) and (A3), one of $\mathfrak{s}(\alpha)$ and $\mathfrak{t}(\alpha)$ has an orbit of size n, so $|G * \alpha| = n$. We can replace β with $\zeta^{-i}\beta$ as the element in $\operatorname{rad}\Lambda/\operatorname{rad}^2\Lambda$ representing the corresponding arrow. By doing this for all n distinct arrows in the orbit of α, we get that on this orbit the action of G coincides with the $*$ action of G. The potential W is not affected by this procedure, if we see it as an element of $\widehat{kQ}/\operatorname{com}_Q$, so it is still invariant under G. However, note that the expression of W as a linear combination of cycles in Q is possibly changed.

In view of the above observation, we can without loss of generality make the additional assumption:

(A6) Arrows with at least one end which is not fixed are sent to arrows by the action of G.

So for an arrow α between two fixed vertices we have $g * \alpha = \alpha = \zeta^{-b(\alpha)} g(\alpha)$, while for all other arrows we have $g * \alpha = g(\alpha) = \beta$ for some arrow $\beta \neq \alpha$.

We need to make some further assumptions about the relationship between G and W. It turns out that it is convenient to impose conditions on the number of fixed vertices appearing in cycles of W. We make the following assumption.

(A7) Every cycle c appearing in W is of one of the following types:

(i) the cycle c goes through no vertices fixed by G;
(ii) the cycle c goes through exactly one (counted with multiplicity) vertex fixed by G;
(iii) the cycle c goes through exactly one (counted with multiplicity) vertex not fixed by G;
(iv) the cycle c goes only through vertices which are fixed by G.

Notation 3.6. We call cycles appearing in W cycles of type (i)–(iv) according to the (mutually exclusive) cases of assumption (A7).

Remark 3.7. These assumptions are strong. We need them to construct a QP (Q_G, W_G) such that the skew group algebra of $\mathcal{P}(Q, W)$ is Morita equivalent to $\mathcal{P}(Q_G, W_G)$. However, the assumptions are satisfied in many examples, and they are weak enough to still hold for (Q_G, W_G). This in turn allows us to come back to (Q, W) via a skew group algebra construction with a natural action of the dual group $\hat{G}$ (see Section 5).

Remark 3.8. From our assumptions, it follows that cycles of a given type are mapped by G to multiples of cycles of the same type. By assumption (A6), cycles of type (i) and (ii) contain only arrows that are mapped to arrows, so those cycles are mapped to cycles. If $c = \alpha_1 \dots \alpha_l$ is of type (iv), then $g(c) = \zeta^{\sum_i b(\alpha_i)} c$, so from $GW = W$ we obtain that

$\sum_i b(\alpha_i) = 0 \pmod{n}$ and $Gc = c$. In particular, $g * c = g(c)$ for all c of type (i), (ii), (iv).

The reader wishing to have examples of QPs with group actions satisfying these assumptions is advised to have in mind the QPs of Example 8.1. In particular, the two QPs of Fig. 4 and Fig. 5 both have an action of $\mathbb{Z}/3\mathbb{Z}$, one sending arrows to arrows and the other multiplying $\tilde{\delta}$ by a third root of unity. They are the quivers with potential corresponding to each other's skew group algebra under these actions. All cycles of the first one are of type (i) or (ii), while all cycles of the second one are of type (iii) or (iv).

3.2. The quiver of ΛG

We now describe the quiver Q_G of the skew group algebra ΛG following [19]. We first define an idempotent $\eta \in \Lambda G$ such that $\eta(\Lambda G)\eta$ is basic and Morita equivalent to ΛG. We decompose η as a sum of primitive orthogonal idempotents, and use those to label the vertices of Q_G. Then we choose elements in $\eta(\Lambda G)\eta$ to be the arrows.

Notation 3.9. A complete list of primitive orthogonal idempotents for the group algebra kG is given by

$$e_\mu = \frac{1}{n} \sum_{i=0}^{n-1} \zeta^{i\mu} g^i,$$

for $\mu = 0, \dots, n-1$.

Notation 3.10. Fix a set $\mathcal{E}$ of representatives of vertices of Q under the action of G. We write $\mathcal{E} = \mathcal{E}' \sqcup \mathcal{E}''$, where $\mathcal{E}'$ and $\mathcal{E}''$ consist of the vertices in $\mathcal{E}$ whose orbits have cardinality n and 1 respectively.

Notation 3.11. We define the following idempotents in ΛG:

- for each vertex $\varepsilon \in \mathcal{E}'$ we put $\eta^\varepsilon = \varepsilon \otimes 1$;
- for each vertex $\varepsilon \in \mathcal{E}''$ and $\mu = 0, \dots, n-1$ we put $\eta^\varepsilon_\mu = \varepsilon \otimes e_\mu$.

Set

$$\eta = \sum_{\varepsilon \in \mathcal{E}'} \eta^\varepsilon + \sum_{\varepsilon \in \mathcal{E}''} \sum_{\mu=0}^{n-1} \eta^\varepsilon_\mu.$$

Note in particular that $\eta = \hat{\varepsilon} \otimes 1$, where $\hat{\varepsilon}$ is the idempotent of Λ corresponding to $\mathcal{E}$. By [19, §2.3] the algebra $\eta(\Lambda G)\eta$ is Morita equivalent to ΛG. A complete list of primitive orthogonal idempotents for $\eta(\Lambda G)\eta$ is

$$\{\eta^\varepsilon \mid \varepsilon \in \mathcal{E}'\} \cup \left\{\eta^\varepsilon_\mu \mid \varepsilon \in \mathcal{E}'', \mu = 0, \dots, n-1\right\}.$$

Remark 3.12. The idempotent η is not canonical, in that it depends on choosing some vertices of Q. However, it is convenient to define it in this way to get a natural action of the dual group $\hat{G}$ on $\eta(\Lambda G)\eta$. By contrast, the authors of [2] choose a canonically defined basic algebra for their Morita equivalence, but in exchange they have to choose vertices of Q in order to be able to define such an action.

Notation 3.13. Now we will fix a basis for the arrows of the quiver Q_G of $\eta(\Lambda G)\eta$. There are four different cases to consider.

(1) Let β be an arrow between two non-fixed vertices of Q. Then there is exactly one arrow α in the G-orbit of β such that $\mathfrak{t}(\alpha) \in \mathcal{E}'$. Thus α is of the form $\alpha\colon g^t\varepsilon \to \varepsilon'$, with $\varepsilon, \varepsilon' \in \mathcal{E}'$ and $0 \le t \le n-1$. We call α an arrow of type (1), and define an element $\tilde{\alpha} \in \eta(\Lambda G)\eta$ by

$$\tilde{\alpha} = \alpha \otimes g^t.$$

This will be an arrow in Q_G from η^ε to $\eta^{\varepsilon'}$.

(2) Let β be an arrow in Q from a non-fixed vertex to a fixed vertex. Then there is exactly one arrow α in the G-orbit of β such that $\mathfrak{s}(\alpha) \in \mathcal{E}'$. Thus α is of the form $\alpha\colon \varepsilon \to \varepsilon'$, with $\varepsilon \in \mathcal{E}'$, $\varepsilon' \in \mathcal{E}''$. We call α an arrow of type (2), and define elements $\tilde{\alpha}^\mu \in \eta(\Lambda G)\eta$ by

$$\tilde{\alpha}^\mu = (1 \otimes e_\mu)(\alpha \otimes 1)$$

for $\mu = 0, \dots, n-1$. These will be arrows in Q_G from η^ε to $\eta^{\varepsilon'}_\mu$ respectively.

(3) Let β be an arrow in Q from a fixed vertex to a non-fixed vertex. Then there is exactly one arrow α in the G-orbit of β such that $\mathfrak{t}(\alpha) \in \mathcal{E}'$. Thus α is of the form $\alpha\colon \varepsilon \to \varepsilon'$, with $\varepsilon \in \mathcal{E}''$, $\varepsilon' \in \mathcal{E}'$. We call α an arrow of type (3), and define elements $\tilde{\alpha}^\mu \in \eta(\Lambda G)\eta$ by

$$\tilde{\alpha}^\mu = \alpha \otimes e_\mu$$

for $\mu = 0, \dots, n-1$. These will be arrows in Q_G from η^ε_μ to $\eta^{\varepsilon'}$ respectively.

(4) Let α be an arrow between two fixed vertices, i.e., $\alpha\colon \varepsilon \to \varepsilon'$ with $\varepsilon, \varepsilon' \in \mathcal{E}''$. Recall that by assumption $g(\alpha) = \zeta^{b(\alpha)}\alpha$. We call α an arrow of type (4), and define elements $\tilde{\alpha}^\mu \in \eta(\Lambda G)\eta$ by

$$\tilde{\alpha}^\mu = \alpha \otimes e_\mu$$

for $\mu = 0, \dots, n-1$. These will be arrows in Q_G from η^ε_μ to $\eta^{\varepsilon'}_{\mu - b(\alpha)}$ respectively.

Remark 3.14. By our construction, not every arrow of Q is of type (1), (2), (3) or (4). However, for each arrow β of Q there exists a unique arrow α of one of those types which is in the G-orbit of β.

Notation 3.15. For an arrow $\alpha : g^t(\varepsilon) \to \varepsilon'$ of type (1), we define $t(\alpha) = t$. Note that this integer is well defined modulo n, since the orbit of ε has cardinality n. If instead α is an arrow of type (2), (3), or (4), we put $t(\alpha) = 0$.

Proposition 3.16. *This choice gives a basis of* $\operatorname{rad}\eta(\Lambda G)\eta/\operatorname{rad}^2\eta(\Lambda G)\eta$, *and the start and target of arrows in* Q_G *are as claimed above.*

Proof. The vector space spanned by the arrows of Q decomposes as a direct sum of kG-modules into the spans of the G-orbits of the arrows. Therefore it is enough to look at one G-orbit of an arrow at a time, and we can assume that there are no multiple arrows in Q.

Let us now look at the four cases. If $\alpha : g^t\varepsilon \to \varepsilon'$ is of type (1), then the n arrows in $G\alpha$ give rise to a unique arrow $\tilde{\alpha} : \eta^{\varepsilon} \to \eta^{\varepsilon'}$. By [19, Theorem 1.3(d)(i)] we have that $\operatorname{rad}^i \Lambda G = (\operatorname{rad}^i \Lambda)\Lambda G$, so that a basis of the space of arrows from η^{ε} to $\eta^{\varepsilon'}$ is given by $\{\varepsilon'\beta h(\varepsilon) \otimes h\}$ with $\beta \in Q_1$. So the only β contributing is the only arrow in $G\alpha$ ending in ε', and this basis is $\{\tilde{\alpha} = \alpha \otimes g^{t(\alpha)}\}$.

Let now $\alpha : \varepsilon \to \varepsilon'$ be of type (2). Then the n arrows in $G\alpha$ give rise to n arrows of Q_G. By the above argument, we get that a basis of $(\varepsilon'\otimes 1)(\operatorname{rad}\Lambda G/\operatorname{rad}^2\Lambda G)(\varepsilon\otimes 1)$ is given by $\{g^i(\alpha) \otimes g^i \mid i = 0, \dots, n-1\}$. Then the set $\{\tilde{\alpha}^{\mu} = (1 \otimes e_{\mu})(\alpha \otimes 1) \mid \mu = 0, \dots, n-1\}$ is also a basis, since

$$(1 \otimes e_{\mu})(\alpha \otimes 1) = \frac{1}{n}\sum_{i=0}^{n-1} \zeta^{i\mu} g^i(\alpha) \otimes g^i.$$

Now $\eta_{\nu}^{\varepsilon'}\tilde{\alpha}^{\mu} = \tilde{\alpha}^{\mu}$ if $\nu = \mu$, and 0 otherwise, so each $\tilde{\alpha}^{\mu}$ is indeed an arrow of Q_G from η^{ε} to $\eta_{\mu}^{\varepsilon'}$.

If $\alpha : \varepsilon \to \varepsilon'$ is an arrow of type (3) or (4), by similar arguments we get that $\{\alpha \otimes g^i\}$ is a basis of $(\varepsilon' \otimes 1)(\operatorname{rad}\Lambda G/\operatorname{rad}^2\Lambda G)(\varepsilon \otimes 1)$. Then $\{\tilde{\alpha}^{\mu} = \alpha \otimes e_{\mu}\}$ is also a basis, and it consists of arrows. $\square$

The choice of vertices and arrows we have made defines an isomorphism $J : kQ_G \to \eta((kQ)G)\eta$ by [19, §2.3].

3.3. Cycles in Q_G and the potential W_G

We want to define a potential W_G on Q_G, so we need to construct cycles in Q_G depending on those appearing in W. Recall that we write $W = \sum_c a(c)c$, and that we consider cycles up to cyclic permutation.

Notation 3.17. We will define, for every cycle c appearing in W, a cycle $\hat{c}$ in $G * c$ depending on our choice of representatives of the vertices. Moreover, to every $\hat{c}$ we will associate a cycle $\tilde{c}$ in Q_G.

(i) Let c be a cycle of type (i) in W. Then choose $\hat{c}$ in $G * c$ such that

$$\hat{c} \quad : \quad \varepsilon_0 = \varepsilon_l \xrightarrow{g^{t_1+\cdots+t_{l-1}}(\alpha_l)} g^{t_1+\cdots+t_{l-1}}(\varepsilon_{l-1}) \longrightarrow \cdots \xrightarrow{g^{t_1}(\alpha_2)} g^{t_1}(\varepsilon_1) \xrightarrow{\alpha_1} \varepsilon_0$$

with $\varepsilon_i \in \mathcal{E}'$ for all i. Notice that this is indeed (in general) a choice, the only requirement is that $\hat{c}$ should go through at least one vertex in $\mathcal{E}'$. Set moreover $\hat{d} = \hat{c}$ for all the other $d \in G * c$. Note that each α_i is an arrow of type (1) and $t_i = t(\alpha_i)$. Define a cycle $\tilde{c}$ in Q_G by

$$\tilde{c} = \tilde{\alpha}_1 \cdots \tilde{\alpha}_l.$$

(ii) Let c be a cycle of type (ii) in W. There is a unique $\hat{c} \in G * c$ that can be written as above, with $\varepsilon_i \in \mathcal{E}'$ for $i \neq 1$, $\varepsilon_1 \in \mathcal{E}''$ and $t_1 = 0$. Note that for $i \geq 3$, α_i is of type (1) and $t_i = t(\alpha_i)$, while $g^{-t_2}(\alpha_2)$ is of type (2) and α_1 is of type (3). Define cycles $\tilde{c}^\mu$ in Q_G by

$$\tilde{c}^\mu = \tilde{\alpha}_1^\mu \widetilde{g^{-t_2}(\alpha_2)}^\mu \tilde{\alpha}_3 \cdots \tilde{\alpha}_l$$

for $\mu = 0, \ldots, n-1$, and call $p(c) = t_2$.

(iii) Let c be a cycle of type (iii) in W. There is a unique $\hat{c} \in G * c$ that can be written as above, with $\varepsilon_i \in \mathcal{E}''$ for $i \neq 1$, $\varepsilon_1 \in \mathcal{E}'$ and $t_i = 0$ for all i. Notice that for $i \geq 3$, α_i is of type (4), while α_2 is of type (3) and α_1 is of type (2). Put $b_i = b(\alpha_i) + \cdots + b(\alpha_l)$ for all $i \geq 3$ and define cycles $\tilde{c}^\mu$ in Q_G by

$$\tilde{c}^\mu = \tilde{\alpha}_1^\mu \tilde{\alpha}_2^{\mu-b_3} \tilde{\alpha}_3^{\mu-b_4} \cdots \tilde{\alpha}_{l-1}^{\mu-b_l} \tilde{\alpha}_l^\mu$$

for $\mu = 0, \ldots, n-1$. Call $q(c) = b_3$, and notice that $g(\hat{c}) = \zeta^{q(c)} g * \hat{c}$ (and in fact $g(c) = \zeta^{q(c)} g * c$).

(iv) Let c be a cycle of type (iv) in W. Thus $Gc = c$ in kQ and we can write $\hat{c} = c$ as above, with $\varepsilon_i \in \mathcal{E}'$ and $t_i = 0$ for all i. Notice that each α_i is an arrow of type (4). Put $b_i = b(\alpha_i) + \cdots + b(\alpha_l)$ for all i and define cycles $\tilde{c}^\mu$ in Q_G by

$$\tilde{c}^\mu = \tilde{\alpha}_1^{\mu-b_2} \tilde{\alpha}_2^{\mu-b_3} \cdots \tilde{\alpha}_{l-1}^{\mu-b_l} \tilde{\alpha}_l^\mu$$

for $\mu = 0, \ldots, n-1$.

Now define $\mathcal{C}(x) = \{\hat{c} \mid c \text{ cycle of } W \text{ of type } x\}$ for $x =$ (i), (ii), (iii), (iv). Then $\mathcal{C} = \bigsqcup \mathcal{C}(x)$ is a cross-section of cycles of W under the $*$ action of G.

Notation 3.18. We can now define a (finite) potential W_G on Q_G by setting

$$W_G = \sum_{c \in \mathcal{C}(\mathrm{i})} a(c) \frac{|Gc|}{n} \tilde{c} + \sum_{c \in \mathcal{C}(\mathrm{ii})} a(c) \sum_{\mu=0}^{n-1} \zeta^{-p(c)\mu} \tilde{c}^{\mu} + \sum_{c \in \mathcal{C}(\mathrm{iii}) \cup \mathcal{C}(\mathrm{iv})} a(c) \sum_{\mu=0}^{n-1} \tilde{c}^{\mu}.$$

Remark 3.19. Note that all cycles in W_G have length at least 3, since each of them has the same length as a cycle in W. Moreover the sums in W_G are made over subsets of cycles which appear in W, hence they are all finite. This means that W_G is indeed a finite potential in Q_G.

3.4. Main result

We are ready to state our main result. Recall that we assume that (Q, W) is a QP with finite potential such that the cyclic derivatives of W generate an admissible ideal of kQ. Call $\Lambda = \mathcal{P}(Q, W)$ the Jacobian algebra of (Q, W).

Theorem 3.20. *Let G be a finite cyclic group acting on (Q, W) as per the assumptions (A1)–(A7). Then*

$$\mathcal{P}(Q_G, W_G) \cong \eta(\Lambda G)\eta.$$

We give a proof of this result in §4.3, and outline here the strategy we will use. By [19, §2.3], the algebra $\eta(\Lambda G)\eta$ is isomorphic to kQ_G modulo a certain ideal. Our first step, carried out in §4.1, is to give explicit generators for this ideal in our setting. However, these generators will not be relations of Q_G (i.e., linear combinations of paths in Q_G with common start and end). In §4.2, we express them in terms of the derivatives of the potential W_G, which will allow us to conclude.

Remark 3.21. The statement that there exists a potential W' such that $\mathcal{P}(Q_G, W') \cong \eta(\Lambda G)\eta$ follows, by taking the 0-th cohomology of the corresponding dg algebras, from a much more general result proved in [14, Corollary 1.3]. Moreover, [14, Lemma 4.4.1] expresses a suitable W' as an element of $\eta(\Lambda G)\eta$, and W' is written as a linear combination of paths in Q_G in the examples of [14, §4.5]. Our Theorem 3.20 states that the potential W_G, which we constructed under our assumptions (A1)–(A7), has the same property.

4. Proof of main result

4.1. Ideals of skew group algebras

In order to prove Theorem 3.20, we need some observations about ideals of skew group algebras.

Proposition 4.1. *Let A be a ring and let η be an idempotent of A. Let $I = AXA$ for some subset $X \subseteq A$, such that $\eta x \eta = x$ for all $x \in X$. Then*

$$\eta \frac{A}{I} \eta = \frac{\eta A \eta}{\langle X \rangle}.$$

Proof. It is enough to prove that $\eta I \eta = \langle X \rangle$. Let $\kappa = 1 - \eta$. Then $\eta A = \eta A \eta \oplus \eta A \kappa$ and $A\eta = \eta A \eta \oplus \kappa A \eta$. Observe that $\eta x \eta = x$ implies $\kappa x = x \kappa = 0$. Then

$$\eta I \eta = \eta AXA\eta = \eta A \eta X \eta A \eta \oplus \eta A \eta X \kappa A \eta \oplus \eta A \kappa X \eta A \eta \oplus \eta A \kappa X \kappa A \eta = \eta A \eta X \eta A \eta = \langle X \rangle. \qquad \square$$

Now retain the notation of Section 3. So $\Lambda = kQ/\mathcal{R}$, where $\mathcal{R} = \langle R \rangle$ and $R = \{\partial_\alpha W \mid \alpha \in Q_1\}$, and the action of G on Λ leaves R stable. Then we know by [19, §2.2] that

$$\Lambda G \cong \frac{(kQ)G}{\langle R \otimes 1 \rangle}.$$

Recall that we have an idempotent $\eta = \hat{\varepsilon} \otimes 1$, for an idempotent $\hat{\varepsilon}$ in kQ, such that $\eta((kQ)G)\eta \cong kQ_G$. We have the following lemmas.

Lemma 4.2. *Suppose that $\langle R \rangle$ is an admissible ideal of kQ. Then the ideal $\eta \langle R \otimes 1 \rangle \eta$ of $\eta((kQ)G)\eta$ is admissible.*

Proof. Let $A = kQ$. Since $\mathcal{R} = \langle R \rangle$ is admissible, we have $(\operatorname{rad} A)^N \subseteq \mathcal{R} \subseteq (\operatorname{rad} A)^2$ for some $N \geq 2$. Consider $\mathcal{R}$ as a subset of AG under the natural inclusion $A \to AG$, so $\langle R \otimes 1 \rangle = (AG)\mathcal{R}(AG)$. By [19, Theorem 1.3(d)(ii)] we have $(AG)(\operatorname{rad} A)^i = (\operatorname{rad} A)^i (AG) = (\operatorname{rad} AG)^i$ for all $i \geq 1$, so

$$(AG)(\operatorname{rad} A)^N (AG) \subseteq (AG)\mathcal{R}(AG) \subseteq (AG)(\operatorname{rad} A)^2 (AG)$$

becomes

$$(\operatorname{rad} AG)^N \subseteq \langle R \otimes 1 \rangle \subseteq (\operatorname{rad} AG)^2.$$

Then the claim follows from the fact that $\eta(\operatorname{rad} AG)\eta = \operatorname{rad}(\eta(AG)\eta)$. $\square$

Lemma 4.3. *For each $r \in R$, choose $g_r, h_r \in G$ such that $\mathfrak{t}(r) \in g_r(\mathcal{E})$ and $\mathfrak{s}(r) \in h_r(\mathcal{E})$. Then*

$$\eta \frac{(kQ)G}{\langle R \otimes 1 \rangle} \eta = \frac{\eta((kQ)G)\eta}{\langle g_r^{-1}(r) \otimes h_r g_r^{-1} \mid r \in R \rangle}.$$

Proof. We have

$$g_r^{-1}(r) \otimes h_r g_r^{-1} = (1 \otimes g_r^{-1})(r \otimes 1)(1 \otimes h_r)$$

so that $R \otimes 1$ generates the same ideal in $(kQ)G$ as the set $\{g_r^{-1}(r) \otimes h_r g_r^{-1} \mid r \in R\}$. Now

$$\eta(g_r^{-1}(r) \otimes h_r g_r^{-1})\eta = \hat{\varepsilon} g_r^{-1}(r)(h_r g_r^{-1})(\hat{\varepsilon}) \otimes h_r g_r^{-1} = g_r^{-1}(r) \otimes h_r g_r^{-1},$$

so the claim follows from Proposition 4.1. □

Lemma 4.4. *In the assumptions (A1)–(A7), we have*

$$\eta(\Lambda G)\eta \cong \frac{\eta((kQ)G)\eta}{\langle \partial_{g^{-t(\alpha)}\alpha} W \otimes g^{-t(\alpha)} \mid \alpha \textit{ of type } (1),(2),(3),(4)\rangle}.$$

Proof. Since G acts on W, the ideal of kQ generated by $\{\partial_\alpha W\} \otimes 1$ is also generated by

$$\{\partial_\alpha W \mid \alpha \text{ of type } (1),(2),(3),(4)\} \otimes 1,$$

since $h(\partial_\alpha W) = \partial_{h(\alpha)} W$ for any $h \in G$. Notice that α is of type (1), (2), (3), (4) precisely if $\mathfrak{s}(\partial_\alpha W) = \mathfrak{t}(\alpha) \in \mathcal{E}$, and then $\mathfrak{t}(\partial_\alpha W) = \mathfrak{s}(\alpha) \in g^{t(\alpha)}(\mathcal{E})$. Then we can apply Lemma 4.3 with $g_r = g^{t(\alpha)}$ and $h_r = 1$, and we get the claim. □

4.2. Derivatives of W_G as elements of $\eta(\Lambda G)\eta$

In this section we shall express elements of the form $\partial_{g^{-t(\alpha)}\alpha} W \otimes g^{-t(\alpha)}$ for α of type (1), (2), (3), (4) in terms of the derivatives of the potential W_G. Precisely, identifying $\eta((kQ)G)\eta$ with kQ_G via the isomorphism J of §3.2, each $\partial_{g^{-t(\alpha)}\alpha} W \otimes g^{-t(\alpha)}$ corresponds to $\sum_{i,j \in (Q_G)_0} x_{ij}$, where x_{ij} is a linear combination of paths in Q_G from vertex i to vertex j (i.e., a relation of Q_G). In Lemma 4.7 we describe the elements x_{ij} in terms of the derivatives of W_G, in a way that depends on the type of α. This will be the last ingredient we need in order to prove Theorem 3.20. We advise the reader to compare Lemma 4.7 with the computations carried out in [14, §4.5].

In the proof of Lemma 4.7, we will use the following identities.

Lemma 4.5. *If $\alpha \in Q_1$ and β is an arrow of type (4), then*

$$(\alpha \otimes e_\mu)(\beta \otimes e_\nu) = \begin{cases} \alpha\beta \otimes e_\nu, & \textit{if } \nu = \mu + b(\beta); \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. We compute

$$(\alpha \otimes e_\mu)(\beta \otimes e_\nu) = \frac{1}{n}\sum_{i=0}^{n-1} \zeta^{i\mu}(\alpha \otimes g^i)(\beta \otimes e_\nu) =$$

$$= \frac{1}{n}\sum_{i=0}^{n-1} \zeta^{i\mu} \alpha g^i(\beta) \otimes g^i e_\nu =$$

$$= \frac{1}{n}\sum_{i=0}^{n-1} \zeta^{i(\mu+b(\beta))}\alpha\beta \otimes g^i e_\nu =$$

$$= \alpha\beta \otimes \frac{1}{n}\sum_{i=0}^{n-1} \zeta^{i(\mu+b(\beta))} g^i e_\nu =$$

$$= \alpha\beta \otimes e_{\mu+b(\beta)} e_\nu$$

and this proves the claim. □

Lemma 4.6. *If c is a cycle of type (iii), then $a(g * c) = \zeta^{q(c)} a(c)$.*

Proof. From assumption (A4), it follows that $g(a(c)c) = a(g * c)g * c$. Then we get the claim since $g(c) = \zeta^{q(c)} g * c$. □

Now we use the identification $kQ_G \cong \eta((kQ)G)\eta$ to see cyclic derivatives of W_G as elements of $\eta((kQ)G)\eta$. To avoid clogging the notation, we will at times write $h\alpha$ and hc instead of $h(\alpha)$ and $h(c)$ for $h \in G$.

Lemma 4.7.

(1) *Let α be an arrow of Q of type (1). Let $\beta = g^{-t(\alpha)}(\alpha)$. Then*

$$\partial_\beta W \otimes g^{-t(\alpha)} = \partial_{\tilde{\alpha}} W_G.$$

(2) *Let α be an arrow of Q of type (2). Then*

$$\partial_\alpha W \otimes 1 = \sum_{\mu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} W_G.$$

In particular,

$$\eta^{\mathfrak{s}(\alpha)}(\partial_\alpha W \otimes 1)\eta_\mu^{\mathfrak{t}(\alpha)} = \partial_{\tilde{\alpha}^\mu} W_G$$

for every $\mu = 0, \dots, n-1$.

(3) *Let α be an arrow of Q of type (3). Then*

$$\partial_\alpha W \otimes 1 = \sum_{\mu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} W_G.$$

In particular,

$$\eta_\mu^{\mathfrak{s}(\alpha)}(\partial_\alpha W \otimes 1)\eta^{\mathfrak{t}(\alpha)} = \partial_{\tilde{\alpha}^\mu} W_G$$

for every $\mu = 0, \dots, n-1$.

(4) *Let α be an arrow of type (4). Then*

$$\partial_\alpha W \otimes 1 = n \sum_{\mu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} W_G.$$

In particular,

$$\eta_\mu^{\mathfrak{s}(\alpha)}(\partial_\alpha W \otimes 1)\eta_{\mu-b(\alpha)}^{\mathfrak{t}(\alpha)} = n\partial_{\tilde{\alpha}^\mu} W_G$$

for every $\mu = 0, \ldots, n-1$.

Proof. First notice that the second part of statements (2), (3), (4) follows directly by multiplying $\sum \partial_{\tilde{\alpha}^\mu} W_G$, which is a linear combination of paths in Q_G, with idempotents corresponding to vertices of Q_G.

It will be convenient to use the following notation: for integers $t_1, \ldots, t_l$, write

$$t_{i,j} = \begin{cases} t_i + t_{i+1} + \cdots + t_j, & \text{if } j \geq i; \\ t_i + t_{i+1} + \cdots + t_l + t_1 + t_2 + \cdots + t_j, & \text{if } j < i. \end{cases}$$

(i) We have that

$$\partial_\beta W \otimes g^{-t(\alpha)} = \sum_{c \text{ of type (i)}} a(c)\partial_\beta c \otimes g^{-t(\alpha)} + \sum_{c \text{ of type (ii)}} a(c)\partial_\beta c \otimes g^{-t(\alpha)}$$

and

$$\partial_{\tilde{\alpha}} W_G = \frac{|Gc|}{n} \sum_{c \in \mathcal{C}(\mathrm{i})} a(c)\partial_{\tilde{\alpha}} \tilde{c} + \sum_{c \in \mathcal{C}(\mathrm{ii})} a(c) \sum_{\mu=0}^{n-1} \zeta^{-p(c)\mu} \partial_{\tilde{\alpha}} \tilde{c}^\mu.$$

The statement will be proved using the following two claims:
Claim (a1). If $c \in \mathcal{C}(\mathrm{i})$, then

$$\sum_{r=0}^{n-1} \partial_\beta g^r c \otimes g^{-t(\alpha)} = \partial_{\tilde{\alpha}} \tilde{c}.$$

Claim (b1). If $c \in \mathcal{C}(\mathrm{ii})$, then

$$\sum_{r=0}^{n-1} \partial_\beta g^r c \otimes g^{-t(\alpha)} = \sum_{\mu=0}^{n-1} \zeta^{-p(c)\mu} \partial_{\tilde{\alpha}^\mu} \tilde{c}^\mu.$$

Assuming these claims hold, let us prove the statement. Recall that by assumption (A6), $gc = g * c$ if c is of type (i) or (ii). We have

$$\sum_{c \text{ of type (i)}} a(c)\partial_\beta c \otimes g^{-t(\alpha)} = \sum_{c\in\mathcal{C}(\mathrm{i})} \sum_{r=0}^{|Gc|-1} a(g^r * c)\partial_\beta(g^r * c) \otimes g^{-t(\alpha)} =$$

$$= \sum_{c\in\mathcal{C}(\mathrm{i})} \frac{|Gc|}{n}\frac{n}{|Gc|} \sum_{r=0}^{|Gc|-1} a(c)\partial_\beta g^r c \otimes g^{-t(\alpha)} =$$

$$= \sum_{c\in\mathcal{C}(\mathrm{i})} \frac{|Gc|}{n} \sum_{r=0}^{n-1} a(c)\partial_\beta g^r c \otimes g^{-t(\alpha)} =$$

$$= \frac{|Gc|}{n} \sum_{c\in\mathcal{C}(\mathrm{i})} a(c)\partial_{\tilde\alpha}\tilde c$$

and

$$\sum_{c \text{ of type (ii)}} a(c)\partial_\beta c \otimes g^{-t(\alpha)} = \sum_{c\in\mathcal{C}(\mathrm{ii})} \sum_{r=0}^{|Gc|-1} a(g^r * c)\partial_\beta(g^r * c) \otimes g^{-t(\alpha)} =$$

$$= \sum_{c\in\mathcal{C}(\mathrm{ii})} \sum_{r=0}^{n-1} a(c)\partial_\beta g^r c \otimes g^{-t(\alpha)} =$$

$$= \sum_{c\in\mathcal{C}(\mathrm{ii})} a(c) \sum_{\mu=0}^{n-1} \zeta^{-p(c)\mu}\partial_{\tilde\alpha^\mu}\tilde c^\mu$$

which together imply that

$$\partial_\beta W \otimes g^{-t(\alpha)} = \partial_{\tilde\alpha} W_G.$$

It remains to prove the claims (a1) and (b1).

Proof of (a1). Since $c \in \mathcal{C}(\mathrm{i})$ we can write

$$c \quad : \quad \varepsilon_0 = \varepsilon_l \xrightarrow{g^{t_{1,l-1}}(\alpha_l)} g^{t_{1,l-1}}(\varepsilon_{l-1}) \longrightarrow \cdots \longrightarrow g^{t_1}(\varepsilon_1) \xrightarrow{\alpha_1} \varepsilon_0.$$

Let $M = \{m \in \{1, \ldots, l\} \mid \alpha = \alpha_m\}$. Then

$$\partial_{\tilde\alpha}\tilde c = \partial_{\tilde\alpha}\tilde\alpha_1 \cdots \tilde\alpha_l = \sum_{m\in M} \tilde\alpha_{m+1}\cdots\tilde\alpha_{m-1} =$$

$$= \sum_{m\in M} (\alpha_{m+1} \otimes g^{t_{m+1}}) \cdots (\alpha_{m-1} \otimes g^{t_{m-1}}) =$$

$$= \sum_{m\in M} \alpha_{m+1} g^{t_{m+1}}(\alpha_{m+2}) \cdots g^{t_{m+1,m-2}}(\alpha_{m-1}) \otimes g^{-t_m}.$$

Note that $t_m = t(\alpha)$ for all $m \in M$, so we are left to prove that

$$\sum_{m\in M}\alpha_{m+1}g^{t_{m+1}}(\alpha_{m+2})\cdots g^{t_{m+1,m-2}}(\alpha_{m-1})=\sum_{r=0}^{n-1}\partial_\beta g^r c.$$

For each $r=0,\dots,n-1$ and $m\in M$, the path $g^r c$ contains the arrow $g^{r+t_{1,m-1}}\alpha_m = g^{r+t_{1,m}}\beta$. Hence, if we define $M_r=\{m\in M\mid r=-t_{1,m}\}$, we have that

$$\partial_\beta g^r c=\sum_{m\in M_r}\alpha_{m+1}g^{t_{m+1}}(\alpha_{m+2})\cdots g^{t_{m+1,m-2}}(\alpha_{m-1}).$$

So the equality we wanted to show becomes

$$\begin{aligned}&\sum_{m\in M}\alpha_{m+1}g^{t_{m+1}}(\alpha_{m+2})\cdots g^{t_{m+1,m-2}}(\alpha_{m-1})\\&\quad=\sum_{r=0}^{n-1}\sum_{m\in M_r}\alpha_{m+1}g^{t_{m+1}}(\alpha_{m+2})\cdots g^{t_{m+1,m-2}}(\alpha_{m-1}),\end{aligned}$$

but this holds because $M=\bigsqcup_{r=0}^{n-1}M_r$.
Proof of (b1). Since $c\in\mathcal{C}(\mathrm{ii})$ we can write

$$c\ :\ \varepsilon_0=\varepsilon_l\xrightarrow{g^{t_{1,l-1}}(\alpha_l)}g^{t_{1,l-1}}(\varepsilon_{l-1})\longrightarrow\cdots\longrightarrow g^{t_2}(\varepsilon_2)\xrightarrow{\alpha_2}\varepsilon_1\xrightarrow{\alpha_1}\varepsilon_0.$$

Recall that by definition $p(c)=t_2$. Let $M=\{m\in\{1,\dots,l\}\mid\alpha=\alpha_m\}$. Then

$$\begin{aligned}\partial_{\tilde\alpha}\tilde c^\mu&=\partial_{\tilde\alpha}\tilde\alpha_1^\mu\widetilde{g^{-p(c)}(\alpha_2)}^{\mu}\tilde\alpha_3\cdots\tilde\alpha_l=\\&=\sum_{m\in M}\tilde\alpha_{m+1}\cdots\tilde\alpha_1^\mu\widetilde{g^{-p(c)}(\alpha_2)}^{\mu}\tilde\alpha_3\cdots\tilde\alpha_{m-1}=\\&=\sum_{m\in M}(\alpha_{m+1}\otimes g^{t_{m+1}})\cdots(\alpha_1\otimes e_\mu)(g^{-t_2}(\alpha_2)\otimes 1)\cdots(\alpha_{m-1}\otimes g^{t_{m-1}}).\end{aligned}$$

Now, recalling that $\sum_{\mu=0}^{n-1}e_\mu=1$ and $\zeta^{-t_2\mu}e_\mu=g^{t_2}e_\mu$, we get

$$\begin{aligned}&\sum_{\mu=0}^{n-1}\zeta^{-t_2\mu}\partial_{\tilde\alpha}\tilde c^\mu\\&\quad=\sum_{m\in M}\sum_{\mu=0}^{n-1}(\alpha_{m+1}\otimes g^{t_{m+1}})\cdots(\alpha_1\otimes\zeta^{-t_2\mu}e_\mu)(g^{-t_2}(\alpha_2)\otimes 1)\cdots(\alpha_{m-1}\otimes g^{t_{m-1}})=\\&\quad=\sum_{m\in M}(\alpha_{m+1}\otimes g^{t_{m+1}})\cdots(\alpha_1\otimes g^{t_2})(g^{-t_2}(\alpha_2)\otimes 1)\cdots(\alpha_{m-1}\otimes g^{t_{m-1}})=\end{aligned}$$

$$= \sum_{m\in M} (\alpha_{m+1}\otimes g^{t_{m+1}})\cdots(\alpha_1\otimes g^{t_1})(\alpha_2\otimes g^{t_2})\cdots(\alpha_{m-1}\otimes g^{t_{m-1}}) =$$
$$= \sum_{m\in M} \alpha_{m+1} g^{t_{m+1}}(\alpha_{m+2})\cdots g^{t_{m+1,m-2}}(\alpha_{m-1})\otimes g^{-t_m}.$$

The rest of the proof of part (b1) is analogous to that of part (a1).

(ii) We have that

$$\partial_\alpha W\otimes 1 = \sum_{c\text{ of type (ii)}} a(c)\partial_\alpha c\otimes 1 + \sum_{c\text{ of type (iii)}} a(c)\partial_\alpha c\otimes 1$$

and

$$\sum_{\mu=0}^{n-1}\partial_{\tilde\alpha^\mu} W_G = \sum_{\mu=0}^{n-1}\sum_{c\in\mathcal{C}(\mathrm{ii})} a(c)\sum_{\nu=0}^{n-1}\zeta^{-p(c)\nu}\partial_{\tilde\alpha^\mu}\tilde c^\nu + \sum_{\mu=0}^{n-1}\sum_{c\in\mathcal{C}(\mathrm{iii})} a(c)\sum_{\nu=0}^{n-1}\partial_{\tilde\alpha^\mu}\tilde c^\nu.$$

The statement will be proved using the following two claims:

Claim (a2). If $c\in\mathcal{C}(\mathrm{ii})$, then $\partial_{\tilde\alpha^\mu}\tilde c^\nu = 0$ for $\mu\neq\nu$ and

$$\sum_{r=0}^{n-1}\partial_\alpha g^r c\otimes 1 = \sum_{\mu=0}^{n-1}\zeta^{-p(c)\mu}\partial_{\tilde\alpha^\mu}\tilde c^\mu.$$

Claim (b2). If $c\in\mathcal{C}(\mathrm{iii})$, then $\partial_{\tilde\alpha^\mu}\tilde c^\nu = 0$ for $\mu\neq\nu$ and

$$\partial_\alpha c\otimes 1 = \sum_{\mu=0}^{n-1}\partial_{\tilde\alpha^\mu}\tilde c^\mu.$$

Assuming these claims hold, let us prove the statement. First notice that if $c\in\mathcal{C}(\mathrm{iii})$ and $\alpha\in h*c$, then $h=1$. We have

$$\sum_{c\text{ of type (ii)}} a(c)\partial_\alpha c\otimes 1 = \sum_{c\in\mathcal{C}(\mathrm{ii})}\sum_{r=0}^{|Gc|-1} a(g^r*c)\partial_\alpha(g^r*c)\otimes 1 =$$
$$= \sum_{c\in\mathcal{C}(\mathrm{ii})}\sum_{r=0}^{n-1} a(c)\partial_\alpha g^r c\otimes 1 =$$
$$= \sum_{c\in\mathcal{C}(\mathrm{ii})} a(c)\sum_{\mu=0}^{n-1}\zeta^{-p(c)\mu}\partial_{\tilde\alpha^\mu}\tilde c^\mu =$$
$$= \sum_{\mu=0}^{n-1}\sum_{c\in\mathcal{C}(\mathrm{ii})} a(c)\sum_{\nu=0}^{n-1}\zeta^{-p(c)\nu}\partial_{\tilde\alpha^\mu}\tilde c^\nu$$

and

$$\begin{aligned}\sum_{c \text{ of type (iii)}} a(c)\partial_\alpha c \otimes 1 &= \sum_{c\in\mathcal{C}(\text{iii})} \sum_{r=0}^{|Gc|-1} a(g^r * c)\partial_\alpha(g^r * c) \otimes 1 = \\ &= \sum_{c\in\mathcal{C}(\text{iii})} a(c)\partial_\alpha c \otimes 1 = \\ &= \sum_{c\in\mathcal{C}(\text{iii})} a(c) \sum_{\mu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} \tilde{c}^\mu = \\ &= \sum_{\mu=0}^{n-1} \sum_{c\in\mathcal{C}(\text{iii})} a(c) \sum_{\nu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} \tilde{c}^\nu\end{aligned}$$

which together imply that

$$\partial_\alpha W \otimes 1 = \sum_{\mu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} W_G.$$

It remains to prove the claims (a2) and (b2).
Proof of (a2). Since $c \in \mathcal{C}(\text{ii})$, we can write it as

$$c \quad : \varepsilon_0 = \varepsilon_l \xrightarrow{g^{t_{1,l-1}}(\alpha_l)} g^{t_{1,l-1}}(\varepsilon_{l-1}) \longrightarrow \cdots \longrightarrow g^{t_2}(\varepsilon_2) \xrightarrow{\alpha_2} \varepsilon_1 \xrightarrow{\alpha_1} \varepsilon_0.$$

If $\alpha \notin g^r c$ for all r then the statement is trivially true. Otherwise, suppose $\alpha \in g^r c$ for some r. Then, since α is of type (2), we necessarily have that $r = -t_2$ and $\alpha = g^{-t_2}(\alpha_2)$ is the only copy of α in $g^{-t_2}c$. Hence

$$\tilde{c}^\nu = \tilde{\alpha}_1^\nu \widetilde{g^{-t_2}(\alpha_2)}^\nu \tilde{\alpha}_3 \cdots \tilde{\alpha}_l = \tilde{\alpha}_1^\nu \tilde{\alpha}^\nu \tilde{\alpha}_3 \cdots \tilde{\alpha}_l$$

and $\partial_{\tilde{\alpha}^\mu} \tilde{c}^\nu = 0$ for $\mu \neq \nu$. We have

$$\begin{aligned}\partial_{\tilde{\alpha}^\mu} \tilde{c}^\mu &= \partial_{\tilde{\alpha}^\mu} \tilde{\alpha}^\mu \tilde{\alpha}_3 \cdots \tilde{\alpha}_l \tilde{\alpha}_1^\mu = \\ &= \tilde{\alpha}_3 \cdots \tilde{\alpha}_l \tilde{\alpha}_1^\mu = \\ &= (\alpha_3 \otimes g^{t_3}) \cdots (\alpha_l \otimes g^{t_l})(\alpha_1 \otimes e_\mu)\end{aligned}$$

so that (recall that $t_{2,l} = 0 \pmod{n}$)

$$\begin{aligned}\sum_{\mu=0}^{n-1} \zeta^{-t_2\mu} \partial_{\tilde{\alpha}^\mu} \tilde{c}^\mu &= \sum_{\mu=0}^{n-1} (\alpha_3 \otimes g^{t_3}) \cdots (\alpha_l \otimes g^{t_l})(\alpha_1 \otimes g^{t_2} e_\mu) = \\ &= \sum_{\mu=0}^{n-1} (\alpha_3 g^{t_3}(\alpha_4) \cdots g^{t_{3,l-1}}(\alpha_l) \otimes g^{-t_2})(\alpha_1 \otimes e_\mu)(1 \otimes g^{t_2}) =\end{aligned}$$

$$
\begin{aligned}
&= \alpha_3 g^{t_3}(\alpha_4)\cdots g^{t_{3,l-1}}(\alpha_l) g^{t_{3,l}}(\alpha_1)\otimes 1 = \\
&= \partial_\alpha g^{-t_2} c \otimes 1 = \\
&= \sum_{r=0}^{n-1} \partial_\alpha g^r c \otimes 1,
\end{aligned}
$$

which proves the claim.

Proof of (b2). We have, since $c \in \mathcal{C}(\text{iii})$,

$$c \quad : \quad \varepsilon_0 = \varepsilon_l \xrightarrow{\alpha_l} \varepsilon_{l-1} \longrightarrow \cdots \longrightarrow \varepsilon_1 \xrightarrow{\alpha_1} \varepsilon_0,$$

with $\alpha = \alpha_1$, and observe that this is the only instance of α in c. Setting $b_i = b(\alpha_i) + \cdots + b(\alpha_l)$ for $i \geq 3$, we have $\tilde{c}^\nu = \tilde{\alpha}^\nu \tilde{\alpha}_2^{\nu - b_3} \tilde{\alpha}_3^{\nu - b_4} \cdots \tilde{\alpha}_l^\nu$, so $\partial_{\tilde{\alpha}^\mu} \tilde{c}^\nu = 0$ for $\mu \neq \nu$. We can compute

$$
\begin{aligned}
\partial_{\tilde{\alpha}^\mu} \tilde{c}^\mu &= \partial_{\tilde{\alpha}^\mu} \tilde{\alpha}^\mu \tilde{\alpha}_2^{\mu - b_3} \tilde{\alpha}_3^{\mu - b_4} \cdots \tilde{\alpha}_l^\mu = \\
&= \tilde{\alpha}_2^{\mu - b_3} \tilde{\alpha}_3^{\mu - b_4} \cdots \tilde{\alpha}_l^\mu = \\
&= (\alpha_2 \otimes e_{\mu - b_3})(\alpha_3 \otimes e_{\mu - b_4}) \cdots (\alpha_l \otimes e_\mu) = \\
&= \alpha_2 \alpha_3 \cdots \alpha_l \otimes e_\mu = \\
&= \partial_\alpha c \otimes e_\mu
\end{aligned}
$$

so that

$$\sum_{\mu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} \tilde{c}^\mu = \partial_\alpha c \otimes 1$$

as claimed.

(iii) We have that

$$\partial_\alpha W \otimes 1 = \sum_{c \text{ of type (ii)}} a(c) \partial_\alpha c \otimes 1 + \sum_{c \text{ of type (iii)}} a(c) \partial_\alpha c \otimes 1$$

and

$$\sum_{\mu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} W_G = \sum_{\mu=0}^{n-1} \sum_{c \in \mathcal{C}(\text{ii})} a(c) \sum_{\nu=0}^{n-1} \zeta^{-p(c)\nu} \partial_{\tilde{\alpha}^\mu} \tilde{c}^\nu + \sum_{\mu=0}^{n-1} \sum_{c \in \mathcal{C}(\text{iii})} a(c) \sum_{\nu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} \tilde{c}^\nu.$$

The statement will be proved using the following two claims:

Claim (a3). If $c \in \mathcal{C}(\text{ii})$, then $\partial_{\tilde{\alpha}^\mu} \tilde{c}^\nu = 0$ for $\mu \neq \nu$ and

$$\partial_\alpha c \otimes 1 = \sum_{\mu=0}^{n-1} \zeta^{-p(c)\mu} \partial_{\tilde{\alpha}^\mu} \tilde{c}^\mu.$$

Claim (b3). If $c \in \mathcal{C}(\mathrm{iii})$, then $\partial_{\tilde{\alpha}^\mu} \tilde{c}^\nu = 0$ for $\mu \neq \nu - q(c)$ and

$$\partial_\alpha c \otimes 1 = \sum_{\mu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} \tilde{c}^{\mu+q(c)}.$$

Assuming these claims hold, let us prove the statement. First notice that if $c \in \mathcal{C}(\mathrm{ii}) \cup \mathcal{C}(\mathrm{iii})$ and $\alpha \in h * c$, then $h = 1$. We have

$$\begin{aligned}\sum_{c \text{ of type (ii)}} a(c)\partial_\alpha c \otimes 1 &= \sum_{c\in\mathcal{C}(\mathrm{ii})} \sum_{r=0}^{|Gc|-1} a(g^r * c)\partial_\alpha(g^r * c) \otimes 1 = \\ &= \sum_{c\in\mathcal{C}(\mathrm{ii})} a(c)\partial_\alpha c \otimes 1 = \\ &= \sum_{c\in\mathcal{C}(\mathrm{ii})} a(c) \sum_{\mu=0}^{n-1} \zeta^{-p(c)\mu} \partial_{\tilde{\alpha}^\mu} \tilde{c}^\mu = \\ &= \sum_{\mu=0}^{n-1} \sum_{c\in\mathcal{C}(\mathrm{ii})} a(c) \sum_{\nu=0}^{n-1} \zeta^{-p(c)\nu} \partial_{\tilde{\alpha}^\mu} \tilde{c}^\nu\end{aligned}$$

and

$$\begin{aligned}\sum_{c \text{ of type (iii)}} a(c)\partial_\alpha c \otimes 1 &= \sum_{c\in\mathcal{C}(\mathrm{iii})} \sum_{r=0}^{|Gc|-1} a(g^r * c)\partial_\alpha(g^r * c) \otimes 1 = \\ &= \sum_{c\in\mathcal{C}(\mathrm{iii})} a(c)\partial_\alpha c \otimes 1 = \\ &= \sum_{c\in\mathcal{C}(\mathrm{iii})} a(c) \sum_{\mu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} \tilde{c}^{\mu+q(c)} = \\ &= \sum_{\mu=0}^{n-1} \sum_{c\in\mathcal{C}(\mathrm{iii})} a(c) \sum_{\nu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} \tilde{c}^\nu\end{aligned}$$

which together imply that

$$\partial_\alpha W \otimes 1 = \sum_{\mu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} W_G.$$

It remains to prove the claims (a3) and (b3).

Proof of (a3). We have, for $c \in \mathcal{C}(\mathrm{ii})$,

$$c \;:\; \varepsilon_0 = \varepsilon_l \xrightarrow{g^{t_{1,l-1}}(\alpha_l)} g^{t_{1,l-1}}(\varepsilon_{l-1}) \longrightarrow \cdots \longrightarrow g^{t_2}(\varepsilon_2) \xrightarrow{\alpha_2} \varepsilon_1 \xrightarrow{\alpha_1} \varepsilon_0,$$

where $\alpha = \alpha_1$, and this is the only copy of α_1 in c. Hence $\tilde{c}^\nu = \tilde{\alpha}^\nu \widetilde{g^{-t_2}(\alpha_2)}^\nu \tilde{\alpha}_3 \cdots \tilde{\alpha}_l$ and $\partial_{\tilde{\alpha}^\mu} \tilde{c}^\nu = 0$ for $\mu \neq \nu$. Then

$$\begin{aligned} \partial_{\tilde{\alpha}^\mu} \tilde{c}^\mu &= \partial_{\tilde{\alpha}^\mu} \tilde{\alpha}^\mu \widetilde{g^{-t_2}(\alpha_2)} \tilde{\alpha}_3 \cdots \tilde{\alpha}_l = \\ &= \widetilde{g^{-t_2}(\alpha_2)} \tilde{\alpha}_3 \cdots \tilde{\alpha}_l = \\ &= (1 \otimes e_\mu)(g^{-t_2}(\alpha_2) \otimes 1)(\alpha_3 \otimes g^{t_3}) \cdots (\alpha_l \otimes g^{t_l}) \end{aligned}$$

and so

$$\begin{aligned} \sum_{\mu=0}^{n-1} \zeta^{-t_2 \mu} \partial_{\tilde{\alpha}^\mu} \tilde{c}^\mu &= \sum_{\mu=0}^{n-1} (1 \otimes e_\mu)(1 \otimes g^{t_2})(g^{-t_2}(\alpha_2) \otimes 1)(\alpha_3 \otimes g^{t_3}) \cdots (\alpha_l \otimes g^{t_l}) = \\ &= (\alpha_2 \otimes g^{t_2})(\alpha_3 \otimes g^{t_3}) \cdots (\alpha_l \otimes g^{t_l}) = \\ &= \partial_\alpha c \otimes 1 \end{aligned}$$

as claimed.

Proof of (b3). We have

$$c \quad : \quad \varepsilon_0 = \varepsilon_l \xrightarrow{\alpha_l} \varepsilon_{l-1} \longrightarrow \cdots \longrightarrow \varepsilon_1 \xrightarrow{\alpha_1} \varepsilon_0,$$

with $\alpha = \alpha_2$, and again observe that this is the only instance of α in c. Write $b_i = b(\alpha_i) + \cdots + b(\alpha_l)$ for $i \geq 3$, and recall that $b_3 = q(c)$. Then $\tilde{c}^\nu = \tilde{\alpha}_1^\nu \tilde{\alpha}^{\nu - b_3} \tilde{\alpha}_3^{\nu - b_4} \cdots \tilde{\alpha}_l^\nu$, so $\partial_{\tilde{\alpha}^\mu} \tilde{c}^\nu = 0$ for $\mu \neq \nu - q(c)$. Hence

$$\begin{aligned} \partial_{\tilde{\alpha}^\mu} \tilde{c}^{\mu + q(c)} &= \partial_{\tilde{\alpha}^\mu} \tilde{\alpha}_1^{\mu + b_3} \tilde{\alpha}^\mu \tilde{\alpha}_3^{\mu + b_3 - b_4} \cdots \tilde{\alpha}_l^{\mu + b_3} = \\ &= \tilde{\alpha}_1^{\mu + b_3} \tilde{\alpha}^\mu \tilde{\alpha}_3^{\mu + b_3 - b_4} \cdots \tilde{\alpha}_l^{\mu + b_3} \tilde{\alpha}_1^{\mu + b_3} = \\ &= (\alpha_3 \otimes e_{\mu + b_3 - b_4}) \cdots (\alpha_l \otimes e_{\mu + b_3})(1 \otimes e_{\mu + b_3})(\alpha_1 \otimes 1) = \\ &= (\alpha_3 \cdots \alpha_l \otimes e_{\mu + b_3})(1 \otimes e_{\mu + b_3})(\alpha_1 \otimes 1) = \\ &= (\alpha_3 \cdots \alpha_l \otimes e_{\mu + b_3})(\alpha_1 \otimes 1) \end{aligned}$$

so

$$\sum_{\mu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} \tilde{c}^{\mu + q(c)} = \alpha_3 \cdots \alpha_l \alpha_1 \otimes 1 = \partial_\alpha c \otimes 1$$

which concludes the proof.

(iv) We have that

$$\partial_\alpha W \otimes 1 = \sum_{c \text{ of type (iii)}} a(c) \partial_\alpha c \otimes 1 + \sum_{c \text{ of type (iv)}} a(c) \partial_\alpha c \otimes 1$$

and

$$n\sum_{\mu=0}^{n-1}\partial_{\tilde{\alpha}^\mu}W_G = n\sum_{\mu=0}^{n-1}\sum_{c\in\mathcal{C}(\mathrm{iii})}a(c)\sum_{\nu=0}^{n-1}\partial_{\tilde{\alpha}^\mu}\tilde{c}^\nu + n\sum_{\mu=0}^{n-1}\sum_{c\in\mathcal{C}(\mathrm{iv})}a(c)\sum_{\nu=0}^{n-1}\partial_{\tilde{\alpha}^\mu}\tilde{c}^\nu.$$

The statement will be proved using the following two claims:
Claim (a4). If $c \in \mathcal{C}(\mathrm{iii})$, then

$$\sum_{r=0}^{n-1}\partial_\alpha g^r c\otimes 1 = \sum_{\mu=0}^{n-1}\sum_{\nu=0}^{n-1}\partial_{\tilde{\alpha}^\mu}\tilde{c}^\nu.$$

Claim (b4). If $c \in \mathcal{C}(\mathrm{iv})$, then

$$\partial_\alpha c\otimes 1 = \sum_{\mu=0}^{n-1}\sum_{\nu=0}^{n-1}\partial_{\tilde{\alpha}^\mu}\tilde{c}^\nu.$$

Assuming these claims hold, let us prove the statement. We have that

$$\begin{aligned}\sum_{c\text{ of type (iii)}}a(c)\partial_\alpha c\otimes 1 &= \sum_{c\in\mathcal{C}(\mathrm{iii})}\sum_{r=0}^{n-1}a(g^r * c)\partial_\alpha(g^r * c)\otimes 1 =\\ &= \sum_{c\in\mathcal{C}(\mathrm{iii})}\sum_{r=0}^{n-1}\zeta^{rq(c)}a(c)\partial_\alpha(g^r * c)\otimes 1 =\\ &= \sum_{c\in\mathcal{C}(\mathrm{iii})}\sum_{r=0}^{n-1}a(c)\partial_\alpha g^r c\otimes 1 =\\ &= n\sum_{\mu=0}^{n-1}\sum_{c\in\mathcal{C}(\mathrm{iii})}a(c)\sum_{\nu=0}^{n-1}\partial_{\tilde{\alpha}^\mu}\tilde{c}^\nu\end{aligned}$$

and

$$\begin{aligned}\sum_{c\text{ of type (iv)}}a(c)\partial_\alpha c\otimes 1 &= \sum_{c\in\mathcal{C}(\mathrm{iv})}\sum_{r=0}^{n-1}a(g^r * c)\partial_\alpha(g^r * c)\otimes 1 =\\ &= \sum_{c\in\mathcal{C}(\mathrm{iv})}\sum_{r=0}^{n-1}a(c)\partial_\alpha c\otimes 1 =\\ &= n\sum_{c\in\mathcal{C}(\mathrm{iv})}a(c)\partial_\alpha c\otimes 1 =\\ &= n\sum_{\mu=0}^{n-1}\sum_{c\in\mathcal{C}(\mathrm{iv})}a(c)\sum_{\nu=0}^{n-1}\partial_{\tilde{\alpha}^\mu}\tilde{c}^\nu\end{aligned}$$

which together imply that

$$\partial_\alpha W \otimes 1 = n \sum_{\mu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} W_G.$$

It remains to prove the claims (a4) and (b4).

Proof of (a4). Let us write, for $c \in \mathcal{C}(\text{iii})$,

$$c \quad : \quad \varepsilon_0 = \varepsilon_l \xrightarrow{\alpha_l} \varepsilon_{l-1} \longrightarrow \cdots \longrightarrow \varepsilon_1 \xrightarrow{\alpha_1} \varepsilon_0,$$

where $\varepsilon_1 \in \mathcal{E}'$ and $\varepsilon_i \in \mathcal{E}''$ for $i \neq 1$.

Let $M = \{m \in \{1, \ldots, l\} \mid \alpha = \alpha_m\}$ and put $b_i = b(\alpha_i) + \cdots + b(\alpha_l)$ for all $i \geq 3$. We have

$$\tilde{c}^\nu \quad : \quad \eta_\nu^{\varepsilon_l} \xrightarrow{\quad \tilde{\alpha}_l^\nu \quad} \eta_{\nu-b_l}^{\varepsilon_{l-1}} \xrightarrow{\quad \tilde{\alpha}_{l-1}^{\nu-b_l} \quad} \cdots \xrightarrow{\quad \tilde{\alpha}_2^{\nu-b_3} \quad} \eta^{\varepsilon_1} \xrightarrow{\tilde{\alpha}_1^\nu} \eta_\nu^{\varepsilon_0},$$

so we may note that, if $m \in M$, the m-th arrow of $\tilde{c}^\nu$ is $\tilde{\alpha}_m^{\nu-b_{m+1}}$, and it coincides with $\tilde{\alpha}^\mu$ if and only if $\nu = \mu + b_{m+1}$. Hence

$$\begin{aligned}
&\sum_{\mu=0}^{n-1}\sum_{\nu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} \tilde{c}^\nu \\
&\quad = \sum_{\mu=0}^{n-1}\sum_{\nu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} \tilde{\alpha}_1^\nu \tilde{\alpha}_2^{\nu-b_3} \tilde{\alpha}_3^{\nu-b_4} \cdots \tilde{\alpha}_{l-1}^{\nu-b_l} \tilde{\alpha}_l^\nu = \\
&\quad = \sum_{\mu=0}^{n-1}\sum_{m\in M} \tilde{\alpha}_{m+1}^{\mu+b_{m+1}-b_{m+2}} \cdots \tilde{\alpha}_l^{\mu+b_{m+1}} \tilde{\alpha}_1^{\mu+b_{m+1}} \tilde{\alpha}_2^{\mu+b_{m+1}-b_3} \cdots \tilde{\alpha}_{m-1}^{\mu+b_{m+1}-b_m} = \\
&\quad = \sum_{\mu=0}^{n-1}\sum_{m\in M} (\alpha_{m+1} \otimes e_{\mu+b_{m+1}-b_{m+2}}) \cdots (\alpha_l \otimes e_{\mu+b_{m+1}}) \\
&\quad (1 \otimes e_{\mu+b_{m+1}})(\alpha_1 \otimes 1)(\alpha_2 \otimes e_{\mu+b_{m+1}-b_3}) \cdots (\alpha_{m-1} \otimes e_{\mu+b_{m+1}-b_m}) = \\
&\quad = \sum_{\mu=0}^{n-1}\sum_{m\in M} (\alpha_{m+1} \cdots \alpha_l \otimes e_{\mu+b_{m+1}})(\alpha_1\alpha_2 \cdots \alpha_{m-1} \otimes e_{\mu+b_{m+1}-b_m}) = \\
&\quad = \sum_{\mu=0}^{n-1}\sum_{m\in M} \frac{1}{n}\sum_{i=0}^{n-1} \zeta^{i(\mu+b_{m+1})} (\alpha_{m+1} \cdots \alpha_l \otimes g^i)(\alpha_1\alpha_2 \cdots \alpha_{m-1} \otimes e_{\mu+b_{m+1}-b_m}) = \\
&\quad = \sum_{\mu=0}^{n-1}\sum_{m\in M} \frac{1}{n}\sum_{i=0}^{n-1} \zeta^{i(\mu+b_{m+1})} \alpha_{m+1} \cdots \alpha_l g^i(\alpha_1\alpha_2 \cdots \alpha_{m-1}) \otimes g^i e_{\mu+b_{m+1}-b_m} =
\end{aligned}$$

$$= \sum_{\mu=0}^{n-1} \sum_{m \in M} \frac{1}{n} \sum_{i=0}^{n-1} \zeta^{ib_m} \alpha_{m+1} \cdots \alpha_l g^i(\alpha_1 \alpha_2 \cdots \alpha_{m-1}) \otimes e_{\mu+b_{m+1}-b_m} =$$

$$= \sum_{\mu=0}^{n-1} \sum_{m \in M} \frac{1}{n} \sum_{i=0}^{n-1} \zeta^{ib_3} \alpha_{m+1} \cdots \alpha_l g^i(\alpha_1 \alpha_2) \cdots \alpha_{m-1} \otimes e_{\mu+b_{m+1}-b_m} =$$

$$= \sum_{\mu=0}^{n-1} \sum_{m \in M} \frac{1}{n} \sum_{i=0}^{n-1} \partial_\alpha g^i c \otimes e_{\mu+b_{m+1}-b_m} =$$

$$= \sum_{m \in M} \frac{1}{n} \sum_{i=0}^{n-1} \partial_\alpha g^i c \otimes 1 =$$

$$= \frac{1}{n} \sum_{i=0}^{n-1} \partial_\alpha g^i c \otimes 1$$

which is what we wanted to prove.
Proof of (b4). Let

$$c \quad : \quad \varepsilon_0 = \varepsilon_l \xrightarrow{\alpha_l} \varepsilon_{l-1} \longrightarrow \cdots \longrightarrow \varepsilon_1 \xrightarrow{\alpha_1} \varepsilon_0,$$

where $\varepsilon_i \in \mathcal{E}''$ for all $i = 1, \ldots, l$. Let $M = \{m \in \{1, \ldots, l\} \mid \alpha = \alpha_m\}$, and put as usual $b_i = b(\alpha_i) + \cdots + b(\alpha_l)$ for all i. We have

$$\tilde{c}^\nu \quad : \quad \eta_\nu^{\varepsilon_l} \xrightarrow{\quad \tilde{\alpha}_l^\nu \quad} \eta_{\nu-b_l}^{\varepsilon_{l-1}} \xrightarrow{\quad \tilde{\alpha}_{l-1}^{\nu-b_l} \quad} \cdots \xrightarrow{\quad \tilde{\alpha}_2^{\nu-b_3} \quad} \eta_{\nu-b_2}^{\varepsilon_1} \xrightarrow{\tilde{\alpha}_1^{\nu-b_2}} \eta_\nu^{\varepsilon_0},$$

hence

$$\begin{aligned}
\sum_{\mu=0}^{n-1} \sum_{\nu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} \tilde{c}^\nu &= \sum_{\mu=0}^{n-1} \sum_{\nu=0}^{n-1} \partial_{\tilde{\alpha}^\mu} \tilde{\alpha}_1^{\nu-b_2} \tilde{\alpha}_2^{\nu-b_3} \cdots \tilde{\alpha}_{l-1}^{\nu-b_l} \tilde{\alpha}_l^\nu = \\
&= \sum_{\mu=0}^{n-1} \sum_{m \in M} \tilde{\alpha}_{m+1}^{\mu+b_{m+1}-b_{m+2}} \cdots \tilde{\alpha}_{m-1}^{\mu+b_{m+1}-b_m} = \\
&= \sum_{\mu=0}^{n-1} \sum_{m \in M} (\alpha_{m+1} \otimes e_{\mu+b_{m+1}-b_{m+2}}) \cdots (\alpha_{m-1} \otimes e_{\mu+b_{m+1}-b_m}) = \\
&= \sum_{\mu=0}^{n-1} \sum_{m \in M} \alpha_{m+1} \cdots \alpha_{m-1} \otimes e_{\mu+b_{m+1}-b_m} = \\
&= \sum_{m \in M} \alpha_{m+1} \cdots \alpha_{m-1} \otimes 1 = \\
&= \partial_\alpha c \otimes 1,
\end{aligned}$$

and the claim is proved. □

4.3. Isomorphism of algebras

We are now ready to prove our main result.

Proof. [Proof of Theorem 3.20] We will first prove that

$$\frac{kQ_G}{\langle \partial_\gamma W_G \mid \gamma \in (Q_G)_1 \rangle} \cong \eta(\Lambda G)\eta.$$

By Lemma 4.4, the right-hand side is isomorphic to

$$\frac{\eta((kQ)G)\eta}{\langle \partial_{g^{-t(\alpha)}\alpha} W \otimes g^{-t(\alpha)} \mid \alpha \text{ of type } (1),(2),(3),(4)\rangle},$$

and by [19, §2.2,§2.3] we have that $kQ_G \cong \eta((kQ)G)\eta$ via the isomorphism J of §3.2. For every arrow α of Q of type (1), (2), (3), (4), we can write (recall that for types (2), (3), (4) we set $t(\alpha) = 0$)

$$J^{-1}\left(\partial_{g^{-t(\alpha)}\alpha} W \otimes g^{-t(\alpha)}\right) = \sum_{i,j \in (Q_G)_0} x_{ij}$$

such that x_{ij} are linear combinations of paths from i to j in kQ_G. By Lemma 4.7, every nonzero x_{ij} is associated in kQ_G to a unique element of the form $\partial_\gamma W_G$ for some $\gamma \in (Q_G)_1$, and moreover every nonzero $\partial_\gamma W_G$ appears in this way for some α. This means that

$$J\left(\langle \partial_\gamma W_G \mid \gamma \in (Q_G)_1 \rangle\right) = \langle \partial_{g^{-t(\alpha)}\alpha} W \otimes g^{-t(\alpha)} \mid \alpha \text{ of type } (1),(2),(3),(4)\rangle$$

so the claim is proved. Now notice that by Lemma 4.2, the ideal $\langle \partial_\gamma W_G \mid \gamma \in (Q_G)_1 \rangle \subseteq kQ_G$ is admissible, so by Proposition 2.2 we conclude that

$$\mathcal{P}(Q_G, W_G) \cong \frac{kQ_G}{\langle \partial_\gamma W_G \mid \gamma \in (Q_G)_1 \rangle}$$

and we are done. □

5. Dual group action

It was proved in [19] that we can always recover the algebra Λ from ΛG by applying another skew group algebra construction. In this section we will show that in our case this construction satisfies again the assumptions (A1)–(A7), and the potential we obtain corresponds to the potential we started with.

Let Λ be a finite dimensional algebra and G be a finite abelian group acting on Λ by automorphisms. We denote by $\hat{G}$ the dual group of G. Its elements are the group homomorphism $\chi\colon G \to k^*$.

Theorem 5.1 *([19, Corollary 5.2]). Define an action of $\hat{G}$ on ΛG by $\chi(\lambda \otimes g) = \chi(g)\lambda \otimes g$, $\lambda \in \Lambda$, $g \in G$. Then the skew group algebra $(\Lambda G)\hat{G}$ is Morita equivalent to Λ.*

We want to apply Theorem 5.1 to our setting, so we retain the notation of Section 3 (in particular we are assuming that $\Lambda = \mathcal{P}(Q, W)$). Since G is finite and cyclic, there is an isomorphism $G \cong \hat{G}$. We can write $\hat{G} = \{\chi_0, \dots, \chi_{n-1}\}$, where we define χ_μ to be the homomorphism which sends g to ζ^μ. Put $\chi = \chi_1$ and note that it is a generator of $\hat{G}$.

Recall that, by Theorem 3.20, we have an isomorphism $\mathcal{P}(Q_G, W_G) \cong \eta(\Lambda G)\eta$, where $\eta \in \Lambda G$ is an idempotent such that $\eta(\Lambda G)\eta$ is Morita equivalent to ΛG and (Q_G, W_G) is the QP described in §3.2.

We will now show that the process of getting back Λ from ΛG is achieved via a construction which satisfies the assumptions (A1)–(A7).

Proposition 5.2. *The dual group $\hat{G}$ acts on $\mathcal{P}(Q_G, W_G)$ by automorphisms and $\mathcal{P}(Q_G, W_G)\hat{G}$ is Morita equivalent to Λ. Moreover this action satisfies the assumptions (A1)–(A7).*

Proof. Since $\eta = \hat{\varepsilon} \otimes 1$ for an idempotent $\hat{\varepsilon} \in \Lambda$, we have that $\hat{G}$ acts trivially on η and so the action of $\hat{G}$ on ΛG restricts to an action on $\eta(\Lambda G)\eta \cong \mathcal{P}(Q_G, W_G)$. Hence, by [19, Lemma 2.2], we have that $(\eta(\Lambda G)\eta)\hat{G}$ is Morita equivalent to $(\Lambda G)\hat{G}$, and the latter is Morita equivalent to Λ by Theorem 5.1. So the first assertion is proved and we are left to check that the action of $\hat{G}$ on (Q_G, W_G) satisfies the assumptions (A1)–(A7).

Assumption (A1) holds because $\hat{G}$ has the same order of G.

If $\varepsilon \in \mathcal{E}'$, then $\chi(\eta^\varepsilon) = \chi(\varepsilon \otimes 1) = \eta^\varepsilon$. If $\varepsilon' \in \mathcal{E}''$ and $0 \le \mu \le n-1$, then

$$\chi(\eta^\varepsilon_\mu) = \chi(\varepsilon \otimes e_\mu) = \frac{1}{n}\sum_{i=0}^{n-1} \zeta^{i\mu}\chi(\varepsilon \otimes g^i) = \frac{1}{n}\sum_{i=0}^{n-1} \zeta^{i(\mu+1)}\varepsilon \otimes g^i = \varepsilon \otimes e_{\mu+1} = \eta^\varepsilon_{\mu+1}.$$

Hence $\hat{G}$ permutes the vertices of Q_G. In particular assumption (A3) holds.

Now we consider the action on the arrows of Q_G. Four cases have to be analysed.

(1) Let α be an arrow of type (1) in Q. Then we have an arrow $\tilde{\alpha} = \alpha \otimes g^{t(\alpha)}$ in Q_G and $\hat{G}$ acts on it as

$$\chi(\tilde{\alpha}) = \chi(\alpha \otimes g^{t(\alpha)}) = \chi(g^{t(\alpha)})\alpha \otimes g^{t(\alpha)} = \zeta^{t(\alpha)}\alpha \otimes g^{t(\alpha)} = \zeta^{t(\alpha)}\tilde{\alpha}.$$

(2) Let α be an arrow of type (2) in Q and $0 \le \mu \le n-1$. Then $\hat{G}$ acts on $\tilde{\alpha}^\mu = (1 \otimes e_\mu)(\alpha \otimes 1)$ as

$$\chi(\tilde{\alpha}^\mu) = \chi((1 \otimes e_\mu)(\alpha \otimes 1)) = (1 \otimes e_{\mu+1})(\alpha \otimes 1) = \tilde{\alpha}^{\mu+1}.$$

(3),(4) Let α be an arrow of type either (3) or (4) in Q and $0 \le \mu \le n-1$. Then $\hat{G}$ acts on $\tilde{\alpha}^\mu = \alpha \otimes e_\mu$ as

$$\chi(\tilde{\alpha}^\mu) = \chi(\alpha \otimes e_\mu) = \alpha \otimes e_{\mu+1} = \tilde{\alpha}^{\mu+1}.$$

This proves assumptions (A2) and (A5).

From these calculations we can deduce how $\hat{G}$ acts on the cycles of W_G. Again we distinguish four cases.

(i) Let c be a cycle of type (i) and write $\tilde{c} = \tilde{\alpha}_1 \cdots \tilde{\alpha}_l$. Then, observing that $t(\alpha_1) + \cdots + t(\alpha_l) = 0 \pmod n$, we get $\chi(\tilde{c}) = \zeta^{t(\alpha_1)+\cdots+t(\alpha_l)}\tilde{c} = \tilde{c}$.

(ii) Let c be a cycle of type (ii) and $0 \le \mu \le n-1$. Write $\tilde{c}^\mu = \tilde{\alpha}_1^\mu \widetilde{g^{-p(c)}(\alpha_2)}^{\mu} \tilde{\alpha}_3 \cdots \tilde{\alpha}_l$. Then we get $\chi(\tilde{c}^\mu) = \zeta^{t(\alpha_3)+\cdots+t(\alpha_l)}\tilde{c}^{\mu+1} = \zeta^{-t(\alpha_2)}\tilde{c}^{\mu+1} = \zeta^{-p(c)}\tilde{c}^{\mu+1}$, since $t(\alpha_1) + \cdots + t(\alpha_l) = 0 \pmod n$ and $t(\alpha_1) = 0$.

(iii) Let c be a cycle of type (iii) and $0 \le \mu \le n-1$. Write $\tilde{c}^\mu = \tilde{\alpha}_1^\mu \tilde{\alpha}_2^\mu \tilde{\alpha}_3^\mu \cdots \tilde{\alpha}_l^\mu$. Then we get $\chi(\tilde{c}^\mu) = \tilde{c}^{\mu+1}$.

(iv) Let c be a cycle of type (iv) and $0 \le \mu \le n-1$. Write $\tilde{c}^\mu = \tilde{\alpha}_1^{\mu-b_2}\tilde{\alpha}_2^{\mu-b_3} \cdots \tilde{\alpha}_{l-1}^{\mu-b_l}\tilde{\alpha}_l^\mu$. Then we get $\chi(\tilde{c}^\mu) = \tilde{c}^{\mu+1}$.

So assumption (A7) is proved.

Finally we get that

$$\begin{aligned}
\chi(W_G) = & \sum_{c \in \mathcal{C}(\mathrm{i})} a(c)\chi(\tilde{c}) + \sum_{c \in \mathcal{C}(\mathrm{ii})} a(c) \sum_{\mu=0}^{n-1} \zeta^{-p(c)\mu}\chi(\tilde{c}^\mu) + \\
& + \sum_{c \in \mathcal{C}(\mathrm{iii})} a(c) \sum_{\mu=0}^{n-1} \chi(\tilde{c}^\mu) + \sum_{c \in \mathcal{C}(\mathrm{iv})} a(c) \sum_{\mu=0}^{n-1} \chi(\tilde{c}^\mu) = \\
= & \sum_{c \in \mathcal{C}(\mathrm{i})} a(c)\tilde{c} + \sum_{c \in \mathcal{C}(\mathrm{ii})} a(c) \sum_{\mu=0}^{n-1} \zeta^{-p(c)(\mu+1)}\tilde{c}^{\mu+1} + \\
& + \sum_{c \in \mathcal{C}(\mathrm{iii})} a(c) \sum_{\mu=0}^{n-1} \tilde{c}^{\mu+1} + \sum_{c \in \mathcal{C}(\mathrm{iv})} a(c) \sum_{\mu=0}^{n-1} \tilde{c}^{\mu+1} = \\
= & W_G,
\end{aligned}$$

so the potential W_G is fixed by $\hat{G}$ and thus assumption (A4) holds. $\square$

To sum up, we have an action of $\hat{G}$ on the Jacobian algebra $\mathcal{P}(Q_G, W_G)$ which satisfies the assumptions (A1)–(A7). Using the procedure described in Section 3 we can construct from it a new QP $((Q_G)_{\hat{G}}, (W_G)_{\hat{G}})$ whose Jacobian algebra is Morita equivalent to Λ. Now we want to construct an explicit isomorphism $\mathcal{P}((Q_G)_{\hat{G}}, (W_G)_{\hat{G}}) \cong \Lambda$.

Firstly, let us give an explicit description of $((Q_G)_{\hat{G}}, (W_G)_{\hat{G}})$.

Let $\mathcal{E}_G = \mathcal{E}'_G \sqcup \mathcal{E}''_G$, where $\mathcal{E}'_G = \{\eta_0^\varepsilon \,|\, \varepsilon \in \mathcal{E}''\}$ and $\mathcal{E}''_G = \{\eta^\varepsilon \,|\, \varepsilon \in \mathcal{E}'\}$. Then $\mathcal{E}_G$ is a set of representatives for the orbits of the action of $\hat{G}$ on Q_G. The elements of $\mathcal{E}'_G$ and $\mathcal{E}''_G$ have orbits of cardinality n and 1 respectively.

The arrows of Q_G can be divided into four families, according to whether their starting and ending points are fixed or not by the action of $\hat{G}$.

(1) Arrows between two non-fixed vertices. These are all the arrows of the form $\tilde{\alpha}^\mu \colon \eta_\mu^\varepsilon \to \eta_{\mu-b(\alpha)}^{\varepsilon'}$, where $\alpha : \varepsilon \to \varepsilon'$ is an arrow of type (4) in Q and $0 \le \mu \le n-1$. Among them, the arrows which are of type (1) with respect to the action of $\hat{G}$ on Q_G are the ones which end in $\mathcal{E}'_G$, i.e., the ones of the form $\tilde{\alpha}^{b(\alpha)} \colon \eta_{b(\alpha)}^\varepsilon \to \eta_0^{\varepsilon'}$. Since $\eta_{b(\alpha)}^\varepsilon = \chi^{b(\alpha)}(\eta_0^\varepsilon)$, we have that $t(\tilde{\alpha}^{b(\alpha)}) = b(\alpha)$.
(2) Arrows from a non-fixed vertex to a fixed one. These are all the arrows of the form $\tilde{\alpha}^\mu \colon \eta_\mu^\varepsilon \to \eta^{\varepsilon'}$, where $\alpha : \varepsilon \to \varepsilon'$ is an arrow of type (3) in Q and $0 \le \mu \le n-1$. Among them, the arrows which are of type (2) with respect to the action of $\hat{G}$ on Q_G are the ones which start in $\mathcal{E}'_G$, i.e., the ones of the form $\tilde{\alpha}^0 \colon \eta_0^\varepsilon \to \eta^{\varepsilon'}$.
(3) Arrows from a fixed vertex to a non-fixed one. These are all the arrows of the form $\tilde{\alpha}^\mu \colon \eta^\varepsilon \to \eta_\mu^{\varepsilon'}$, where $\alpha : \varepsilon \to \varepsilon'$ is an arrow of type (2) in Q and $0 \le \mu \le n-1$. Among them, the arrows which are of type (3) with respect to the action of $\hat{G}$ on Q_G are the ones which end in $\mathcal{E}'_G$, i.e., the ones of the form $\tilde{\alpha}^0 \colon \eta^\varepsilon \to \eta_0^{\varepsilon'}$.
(4) Arrows between two fixed vertices. These are all the arrows of the form $\tilde{\alpha} \colon \eta^\varepsilon \to \eta^{\varepsilon'}$, where $\alpha : \varepsilon \to \varepsilon'$ is an arrow of type (1) in Q. All of them are of type (4) with respect to the action of $\hat{G}$ on Q_G. Since $\chi(\tilde{\alpha}) = \zeta^{t(\alpha)}\tilde{\alpha}$, we have that $b(\tilde{\alpha}) = t(\alpha)$.

We deduce that the quiver $(Q_G)_{\hat{G}}$ is made as follows. Its vertices are $\eta_0^\varepsilon \otimes 1$ for $\varepsilon \in \mathcal{E}''$ and $\eta^\varepsilon \otimes e_\nu$ for $\varepsilon \in \mathcal{E}'$, $0 \le \nu \le n-1$, while its arrows are the following:

(1) $\tilde{\beta} \colon \eta_0^\varepsilon \otimes 1 \to \eta_0^{\varepsilon'} \otimes 1$, where $\beta = \tilde{\alpha}^{b(\alpha)}$ and $\alpha : \varepsilon \to \varepsilon'$ is an arrow of type (4) in Q,
(2) $\tilde{\beta}^\nu \colon \eta_0^\varepsilon \otimes 1 \to \eta^{\varepsilon'} \otimes e_\nu$, where $\beta = \tilde{\alpha}^0$, $0 \le \nu \le n-1$ and $\alpha : \varepsilon \to \varepsilon'$ is an arrow of type (3) in Q,
(3) $\tilde{\beta}^\nu \colon \eta^\varepsilon \otimes e_\nu \to \eta_0^{\varepsilon'} \otimes 1$, where $\beta = \tilde{\alpha}^0$, $0 \le \nu \le n-1$ and $\alpha : \varepsilon \to \varepsilon'$ is an arrow of type (2) in Q,
(4) $\tilde{\beta}^\nu \colon \eta^\varepsilon \otimes e_\nu \to \eta^{\varepsilon'} \otimes e_{\nu-t(\alpha)}$, where $\beta = \tilde{\alpha}$, $0 \le \nu \le n-1$ and $\alpha : \varepsilon \to \varepsilon'$ is an arrow of type (1) in Q.

Proposition 5.3. *Let $\phi \colon (Q_G)_{\hat{G}} \to Q$ be the morphism of quivers defined as follows.*

- $\phi(\eta_0^\varepsilon \otimes 1) = \varepsilon$ *for* $\varepsilon \in \mathcal{E}''$.
- $\phi(\eta^\varepsilon \otimes e_\mu) = g^\mu(\varepsilon)$ *for* $\varepsilon \in \mathcal{E}'$, $0 \le \mu \le n-1$.
- $\phi(\tilde{\beta}) = \alpha$, *where* $\beta = \tilde{\alpha}^{b(\alpha)}$ *and* α *is an arrow of type (4) in* Q.

- *$\phi(\tilde{\beta}^\nu) = g^\nu(\alpha)$, where $\beta = \tilde{\alpha}^0$, $0 \le \nu \le n-1$ and α is an arrow of type (3) in Q.*
- *$\phi(\tilde{\beta}^\nu) = g^\nu(\alpha)$, where $\beta = \tilde{\alpha}^0$, $0 \le \nu \le n-1$ and α is an arrow of type (2) in Q.*
- *$\phi(\tilde{\beta}^\nu) = g^{\nu - t(\alpha)}(\alpha)$, where $\beta = \tilde{\alpha}$, $0 \le \nu \le n-1$ and α is an arrow of type (1) in Q.*

Then ϕ is an isomorphism and, if we extend it to an isomorphism between the corresponding path algebras, we have $\phi((W_G)_{\hat{G}}) = W$.

Proof. We first note that ϕ is a well defined morphism of quivers. Moreover, by what we observed earlier in this section, ϕ is a bijection on both the sets of vertices and arrows, thus it is an isomorphism.

Given the set $\mathcal{E}_G$ defined above, we can choose a set $\mathcal{C}_G = \{\hat{d} \,|\, d \text{ cycle in } W_G\}$ of representatives for the $*$ action of $\hat{G}$ on cycles as in §3.3. We have that $\mathcal{C}_G = \mathcal{C}_G(\mathrm{i}) \sqcup \mathcal{C}_G(\mathrm{ii}) \sqcup \mathcal{C}_G(\mathrm{iii}) \sqcup \mathcal{C}_G(\mathrm{iv})$. We now describe each of these four subsets and show where their elements are sent by ϕ. We use the notation $t_{i,j}$ of the proof of Lemma 4.7.

(i) Cycles of type (i) in Q_G are the ones of the form $d = \tilde{c}^\mu$, where $c \in \mathcal{C}(\mathrm{iv})$. If we write $c = \alpha_1 \cdots \alpha_l$ for some arrows α_i of type (4) in Q, then $\tilde{c}^\mu = \tilde{\alpha}_1^{\mu-b_2}\tilde{\alpha}_2^{\mu-b_3}\tilde{\alpha}_3^{\mu-b_4}\cdots\tilde{\alpha}_{l-1}^{\mu-b_l}\tilde{\alpha}_l^{\mu}$, where $b_i = b(\alpha_i) + \cdots + b(\alpha_l)$. Hence we can choose $\hat{d} = \tilde{\alpha}_1^{-b_2}\tilde{\alpha}_2^{-b_3}\tilde{\alpha}_3^{-b_4}\cdots\tilde{\alpha}_{l-1}^{-b_l}\tilde{\alpha}_l^{0} = \tilde{c}^0$, and $\mathcal{C}_G(\mathrm{i})$ is the subset of all the cycles of this kind. Moreover we have that $\tilde{d} = \tilde{\beta}_1 \cdots \tilde{\beta}_l$, where $\beta_i = \tilde{\alpha}_i^{b(\alpha_i)}$. It follows that

$$\phi(\tilde{d}) = \phi(\tilde{\beta}_1 \cdots \tilde{\beta}_l) = \alpha_1 \cdots \alpha_l = c.$$

Let us now look at the coefficient $a(d)$ of d as a summand of W_G. The cycle c of W gives rise to a number $x = |\hat{G}\tilde{c}^\mu|$ of distinct cycles in W_G (this does not depend on the choice of μ). Then $a(d) = a(c)\frac{n}{x}$.

(ii) Cycles of type (ii) in Q_G are the ones of the form $d = \tilde{c}^\mu$, where $c \in \mathcal{C}(\mathrm{iii})$. If we write $c = \alpha_1\alpha_2\cdots\alpha_l$ for α_1 of type (2), α_2 of type (3), and $\alpha_3, \ldots, \alpha_l$ of type (4) in Q, then $\tilde{c}^\mu = \tilde{\alpha}_1^{\mu}\tilde{\alpha}_2^{\mu-b_3}\tilde{\alpha}_3^{\mu-b_4}\cdots\tilde{\alpha}_{l-1}^{\mu-b_l}\tilde{\alpha}_l^{\mu}$, where we write $b_i = b(\alpha_i) + \cdots + b(\alpha_l)$. Hence we obtain that $\hat{d} = \tilde{\alpha}_1^{0}\tilde{\alpha}_2^{-b_3}\tilde{\alpha}_3^{-b_4}\cdots\tilde{\alpha}_{l-1}^{-b_l}\tilde{\alpha}_l^{0} = \tilde{c}^0$, and $\mathcal{C}_G(\mathrm{ii})$ is the subset of all the cycles of this kind. Moreover we have that $\tilde{d}^\nu = \tilde{\beta}_1^\nu\tilde{\beta}_2^\nu\tilde{\beta}_3\cdots\tilde{\beta}_l$, where $\beta_1 = \tilde{\alpha}_1^0$, $\beta_2 = \tilde{\alpha}_2^0$ and $\beta_i = \tilde{\alpha}_i^{b(\alpha_i)}$ for $i \ge 3$. It follows that (recall that by definition $q(c) = b_3$)

$$\phi(\tilde{d}) = \phi(\tilde{\beta}_1^\nu\tilde{\beta}_2^\nu\tilde{\beta}_3\cdots\tilde{\beta}_l) = g^\nu(\alpha_1)g^\nu(\alpha_2)\alpha_3\cdots\alpha_l = \zeta^{-b_3}g^\nu(c) = \zeta^{-q(c)}g^\nu(c).$$

Note that $\beta_2 = \chi^{b_3}\tilde{\alpha}_2^{-b_3}$. This implies that $p(d) = -q(c)$ and so $\phi(\tilde{d}) = \zeta^{p(d)}g^\nu(c)$.

(iii) Cycles of type (iii) in Q_G are the ones of the form $d = \tilde{c}^\mu$, where $c \in \mathcal{C}(\mathrm{ii})$. If we write $c = \alpha_1\alpha_2 g^{t_2}(\alpha_3)\cdots g^{t_{2,l-1}}(\alpha_l)$ for α_1 of type (3), α_2 of type (2), and $\alpha_3, \ldots, \alpha_l$ of type (1) in Q, then $\tilde{c}^\mu = \tilde{\alpha}_1^\mu \widetilde{g^{-t_2}(\alpha_2)}^{\mu}\tilde{\alpha}_3\cdots\tilde{\alpha}_l$. Hence $\hat{d} = \tilde{\alpha}_1^0 \widetilde{g^{-t_2}(\alpha_2)}^{0}\tilde{\alpha}_3\cdots\tilde{\alpha}_l = \tilde{c}^0$, and $\mathcal{C}_G(\mathrm{iii})$ is the subset of all the cycles of this kind. Now define $\beta_1 = \tilde{\alpha}_1^0$, $\beta_2 = \widetilde{g^{-t_2}(\alpha_2)}^{0}$ and $\beta_i = \tilde{\alpha}_i$ for $i \ge 3$. Recall that, for $i \ge 3$, $\chi(\beta_i) = \zeta^{t(\alpha_i)}\beta_i$, so $b(\beta_i) = t(\alpha_i)$. If we put $b'_i = b(\beta_i) + \cdots + b(\beta_l)$ for $i \geq 3$, we have that $\tilde{d}^\nu = \tilde{\beta}_1^\nu \tilde{\beta}_2^{\nu-b'_3} \tilde{\beta}_3^{\nu-b'_4} \cdots \tilde{\beta}_{l-1}^{\nu-b'_l} \tilde{\beta}_l^\nu$. Then

$$\begin{aligned}
\phi(\tilde{d}^\nu) &= \phi(\tilde{\beta}_1^\nu \tilde{\beta}_2^{\nu-b'_3} \tilde{\beta}_3^{\nu-b'_4} \cdots \tilde{\beta}_{l-1}^{\nu-b'_l} \tilde{\beta}_l^\nu) = \\
&= g^\nu(\alpha_1) g^{\nu-b'_3}(g^{-t_2}(\alpha_2)) g^{\nu-b'_4-t(\alpha_3)}(\alpha_3) \cdots g^{\nu-t(\alpha_l)}(\alpha_l) = \\
&= g^\nu(\alpha_1 g^{-t_{2,l}}(\alpha_2) g^{-t_{3,l}}(\alpha_3) \cdots g^{-t_l}(\alpha_l)) = \\
&= g^\nu(\alpha_1 \alpha_2 g^{t_2}(\alpha_3) \cdots g^{t_{2,l-1}}(\alpha_l)) = \\
&= g^\nu(c).
\end{aligned}$$

(iv) Cycles of type (iv) in Q_G are the ones of the form $d = \tilde{c}$, where $c \in \mathcal{C}(\mathrm{i})$. If we write $c = \alpha_1 g^{t_1}(\alpha_2) g^{t_{1,2}}(\alpha_3) \cdots g^{t_{1,l-1}}(\alpha_l)$ for α_i of type (1) in Q, then $\tilde{c} = \tilde{\alpha}_1 \cdots \tilde{\alpha}_l$. Hence $\hat{d} = d$, and $\mathcal{C}_G(\mathrm{iv})$ is the subset of all the cycles of this kind. If we put $\beta_i = \tilde{\alpha}_i$ for all i, then $\tilde{d}^\nu = \tilde{\beta}_1^{\nu-b'_2} \tilde{\beta}_2^{\nu-b'_3} \tilde{\beta}_3^{\nu-b'_4} \cdots \tilde{\beta}_{l-1}^{\nu-b'_l} \tilde{\beta}_l^\nu$. It follows that

$$\begin{aligned}
\phi(\tilde{d}^\nu) &= \phi(\tilde{\beta}_1^{\nu-b'_2} \tilde{\beta}_2^{\nu-b'_3} \tilde{\beta}_3^{\nu-b'_4} \cdots \tilde{\beta}_{l-1}^{\nu-b'_l} \tilde{\beta}_l^\nu) = \\
&= g^{\nu-b'_2-t(\alpha_1)}(\alpha_1) g^{\nu-b'_3-t(\alpha_2)}(\alpha_2) \cdots g^{\nu-t(\alpha_l)}(\alpha_l) = \\
&= g^\nu(\alpha_1 g^{t_1}(\alpha_2) g^{t_{1,2}}(\alpha_3) \cdots g^{t_{1,l-1}}(\alpha_l)) = \\
&= g^\nu(c).
\end{aligned}$$

Now we can write $(W_G)_{\hat{G}}$ as follows:

$$\begin{aligned}
(W_G)_{\hat{G}} = &\sum_{d \in \mathcal{C}_G(\mathrm{i})} a(d) \frac{|\hat{G}d|}{n} \tilde{d} + \sum_{d \in \mathcal{C}_G(\mathrm{ii})} a(d) \sum_{\nu=0}^{n-1} \zeta^{-p(d)\nu} \tilde{d}^\nu + \\
&+ \sum_{d \in \mathcal{C}_G(\mathrm{iii})} a(d) \sum_{\nu=0}^{n-1} \tilde{d}^\nu + \sum_{d \in \mathcal{C}_G(\mathrm{iv})} a(d) \sum_{\nu=0}^{n-1} \tilde{d}^\nu = \\
&= \sum_{c \in \mathcal{C}(\mathrm{iv}), d = \tilde{c}^0} a(c) \tilde{d} + \sum_{c \in \mathcal{C}(\mathrm{iii}), d = \tilde{c}^0} a(c) \sum_{\nu=0}^{n-1} \zeta^{q(c)\nu} \tilde{d}^\nu + \\
&+ \sum_{c \in \mathcal{C}(\mathrm{ii}), d = \tilde{c}^0} a(c) \sum_{\nu=0}^{n-1} \tilde{d}^\nu + \sum_{c \in \mathcal{C}(\mathrm{i}), d = \tilde{c}} a(c) \frac{|Gc|}{n} \sum_{\nu=0}^{n-1} \tilde{d}^\nu.
\end{aligned}$$

Applying ϕ we get

$$\begin{aligned}
\phi((W_G)_{\hat{G}}) = &\sum_{c\in\mathcal{C}(\mathrm{iv}),d=\tilde{c}^0} a(c)\phi(\tilde{d}) + \sum_{c\in\mathcal{C}(\mathrm{iii}),d=\tilde{c}^0} a(c)\sum_{\nu=0}^{n-1}\zeta^{q(c)\nu}\phi(\tilde{d}^\nu)+\\
&+\sum_{c\in\mathcal{C}(\mathrm{ii}),d=\tilde{c}^0} a(c)\sum_{\nu=0}^{n-1}\phi(\tilde{d}^\nu) + \sum_{c\in\mathcal{C}(\mathrm{i}),d=\tilde{c}} a(c)\frac{|Gc|}{n}\sum_{\nu=0}^{n-1}\phi(\tilde{d}^\nu) =\\
&= \sum_{c\in\mathcal{C}(\mathrm{iv}),d=\tilde{c}^0} a(c)c + \sum_{c\in\mathcal{C}(\mathrm{iii}),d=\tilde{c}^0} a(c)\sum_{\nu=0}^{n-1}\zeta^{q(c)\nu}\zeta^{-q(c)\nu}g^\nu(c)+\\
&+\sum_{c\in\mathcal{C}(\mathrm{ii}),d=\tilde{c}^0} a(c)\sum_{\nu=0}^{n-1}g^\nu(c) + \sum_{c\in\mathcal{C}(\mathrm{i}),d=\tilde{c}} a(c)\frac{|Gc|}{n}\sum_{\nu=0}^{n-1}g^\nu(c) =\\
&= \sum_{c\in\mathcal{C}(\mathrm{iv})} a(c)c + \sum_{\nu=0}^{n-1} g^\nu\left(\sum_{c\in\mathcal{C}(\mathrm{iii})} a(c)c + \sum_{c\in\mathcal{C}(\mathrm{ii})} a(c)c + \sum_{c\in\mathcal{C}(\mathrm{i})} a(c)\frac{|Gc|}{n}c\right) =\\
&= W. \qquad \square
\end{aligned}$$

Corollary 5.4. *Let θ be the idempotent $\sum_{s\in\mathcal{E}_G} s\otimes 1$ in $(\eta(\Lambda G)\eta)\hat{G}$. Then the isomorphism of quivers $\phi : (Q_G)_{\hat{G}} \to Q$ induces an isomorphism of algebras*

$$\theta\left((\eta\,(\Lambda G)\,\eta)\,\hat{G}\right)\theta \cong \Lambda,$$

where $\Lambda = \mathcal{P}(Q, W)$.

Proof. Applying Theorem 3.20 to $\eta(\Lambda G)\eta$ with the action of $\hat{G}$, we get

$$\theta\left((\eta\,(\Lambda G)\,\eta)\,\hat{G}\right)\theta \cong \mathcal{P}((Q_G)_{\hat{G}}, (W_G)_{\hat{G}}),$$

and the latter is isomorphic to $\mathcal{P}(Q, W)$ by Proposition 5.3. $\square$

6. Planar rotation-invariant QPs

Our main result Theorem 3.20 is about skew group algebras of Jacobian algebras of QPs, but it only applies under some assumptions on the group action. There is however a class of QPs which satisfy these assumptions, as well as a way of generating many examples in this class. To define this class, we follow [8] and associate a CW-complex to a QP called its canvas. First we need to fix some notation.

We denote by D^d the d-disk and by $S^{d-1} = \partial D^d$ the $(d-1)$-sphere in $\mathbb{R}^d$. We suppose that $D^1 = [0, 1]$ and $S^0 = \{0, 1\}$. A CW-complex is a topological space realized as a union $\bigcup_{d\in\mathbb{Z}_{\geq 0}} X^d$, where X^0 is a discrete space and each X^d is obtained from X^{d-1} in the following way. For each d there are a set $\{D^d_a\}_{a\in I_d}$ of copies of the d-disk and continuous maps $\phi_a\colon S^{d-1}_a = \partial D^d_a \to X^{d-1}$, such that we have a pushout diagram

$$\begin{tikzcd} \bigsqcup_{a\in I_d} S_a^{d-1} \arrow[r, "(\phi_a)"] \arrow[d] & X^{d-1} \arrow[d] \\ \bigsqcup_{a\in I_d} D_a^{d} \arrow[r, "(\epsilon_a)"] & X^{d} \end{tikzcd}$$

in the category of topological spaces with continuous maps (the left vertical map is given by the inclusions of S_a^{d-1} as boundaries of D_a^d). For $d \geq 1$ the image of the interior of D_a^d under ϵ_a is called a d-cell. The elements of X_0 are called 0-cells. We say that X has dimension m if $X = X^m$.

Definition 6.1 *([8, Definition 8.1]).* Let (Q, W) be a QP and let Q_2 be a set of representatives modulo com_Q of the cycles which appear in W. The *canvas* of (Q, W) is the 2-dimensional CW-complex $X_{(Q,W)}$ defined in the following way. Its cells are indexed by the sets $X_0 = Q_0$, $I_1 = Q_1$, $I_2 = Q_2$. For each $\alpha \in Q_1$ we have an attaching map $\phi_\alpha\colon S_\alpha^0 \to X_0$ defined by $\phi_\alpha(0) = \mathfrak{s}(\alpha)$ and $\phi_\alpha(1) = \mathfrak{t}(\alpha)$. If $c = \alpha_0 \cdots \alpha_{l-1} \in Q_2$, we define the attaching map $\phi_c\colon S_c^1 \to X_1$ by

$$\phi_c \left(\cos \left(\frac{2\pi}{l}(i+t) \right), \sin \left(\frac{2\pi}{l}(i+t) \right) \right) = \epsilon_{\alpha_i}(t)$$

for $i = 0, \ldots, l-1$ and $t \in [0, 1)$.

Remark 6.2. In other (imprecise) words, the 1-skeleton of $X_{(Q,W)}$ is the underlying graph of Q, and we attach 2-cells along the cycles appearing in W.

Definition 6.3 *([8, Definition 9.1]).* A QP (Q, W) is *planar* if it is simply connected and there exists an embedding of $X_{(Q,W)}$ into $\mathbb{R}^2$. We call it *strongly planar* if it is planar and $X_{(Q,W)}$ is homeomorphic to a disk.

If (Q, W) is a planar QP, then by [8, Proposition 9.3] the embedding of the quiver Q in $\mathbb{R}^2$ determines the Jacobian algebra, so we can assume that the coefficients in W are $+1$ for the clockwise faces, and -1 for the anticlockwise faces.

Definition 6.4. Let (Q, W) be a planar QP and G be a cyclic group acting on Q. We say that G *acts on* (Q, W) *by rotations* if:

- there is an embedding of $X_{(Q,W)}$ in $\mathbb{R}^2$ such that the action of a generator of G is induced by a rotation of the plane;
- the action of G is faithful;
- assumption (A7) is satisfied.

Notice that in this case the image $\mathrm{im}(G) \subseteq \mathrm{Aut}(Q)$ is necessarily finite. For simplicity, we will identify G with $\mathrm{im}(G)$.

We remark some facts which follow immediately from the definition, and directly imply that this class of quivers falls within the scope of Theorem 3.20.

Lemma 6.5. *Let G act on a planar QP (Q, W) by rotations. Then the action of G satisfies the assumptions (A2)–(A7).*

Proof. A rotation permutes the vertices and maps arrows to arrows, so assumptions (A2) and (A6) are satisfied. By Remark 3.3, we can assume that assumption (A5) is also satisfied. Since we are assuming that G acts faithfully, we have that every vertex which is not fixed has order the order of a rotation generating G, hence assumption (A3) is satisfied. Assumption (A4) holds because G maps faces of $X_{(Q,W)}$ to faces. Finally, assumption (A7) holds by definition. □

There is a way of producing strongly planar QPs with a group acting by rotations by means of so-called Postnikov diagrams (see [17], [3], [16]). A Postnikov diagram is a collection of oriented curves in a disk subject to some axioms depending on two integer parameters $a, n \geq 1$, and it naturally gives rise to a planar QP. For this result we need to assume that $k = \mathbb{C}$.

Theorem 6.6 *([16, Corollary 7.3]). An (a, n)-Postnikov diagram is invariant under rotation by $\frac{2\pi a}{n}$ if and only if the corresponding QP is self-injective. In this case, a Nakayama automorphism is given by this rotation.*

In particular, there is a finite cyclic group acting by rotations on a planar QP, so we can apply our construction. The following result justifies the claim that Postnikov diagrams give rise to many examples. Namely, rotation-invariant Postnikov diagrams exist and in fact abound.

Theorem 6.7. *[18] There exists an (a, n)-Postnikov diagram which is invariant under rotation by $\frac{2\pi a}{n}$ if and only if a is congruent to -1, 0 or 1 modulo $n/\operatorname{GCD}(n, a)$. In particular there are infinitely many self-injective planar QPs with Nakayama automorphism of order d, for any choice of d.*

Remark 6.8. There exist self-injective planar QPs with Nakayama automorphism acting by rotation which do not come from Postnikov diagrams. For instance, the quiver of the 3-preprojective algebra of type A_n (see Example 8.1) with n odd.

We conclude this section by observing that Theorem 3.20 can be naturally applied to any self-injective QP where the Nakayama automorphism satisfies our assumptions. In this case we get:

Proposition 6.9. *Let (Q, W) be a self-injective QP with Nakayama automorphism φ of finite order. Call $G = \langle\varphi\rangle \subseteq \mathrm{Aut}(\mathcal{P}(Q, W))$, and assume that the assumptions (A1)–(A7) are satisfied. Then $\mathcal{P}(Q_G, W_G)$ is symmetric.*

Proof. By Theorem 3.20, $\mathcal{P}(Q_G, W_G)$ is a self-injective algebra which is Morita equivalent to ΛG. The latter is symmetric by Corollary 2.6 using Lemma 2.7. □

Combining this with our previous discussion, we remark that by Theorem 6.6 there is a symmetric Jacobian algebra associated to every rotation-invariant Postnikov diagram.

Corollary 6.10. *If (Q, W) is a self-injective QP coming from a Postnikov diagram with Nakayama automorphism φ, then $\mathcal{P}(Q_{\langle\varphi\rangle}, W_{\langle\varphi\rangle})$ is symmetric.*

These results are illustrated in Example 8.3.

7. Cuts and 2-representation finite algebras

In this section we apply our construction to the study of 2-representation finite algebras. These are by definition algebras of global dimension at most 2 admitting a cluster tilting module, and were introduced by Iyama as a natural generalisation of hereditary representation finite algebras. We refer the interested reader to [12], [13] for general higher Auslander–Reiten theory, and to [8] for the 2-dimensional case. For the general interaction between higher representation finiteness and skew group algebras, see also [14].

Let (Q, W) be a QP. For a subset $C \subseteq Q_1$ we can define a grading d_C on Q by setting

$$d_C(\alpha) = \begin{cases} 1, & \text{if } \alpha \in C; \\ 0, & \text{otherwise.} \end{cases}$$

Definition 7.1. A subset $C \subseteq Q_1$ is called a *cut* if W is homogeneous of degree 1 with respect to d_C.

Note that a cut induces a grading on the Jacobian algebra $\mathcal{P}(Q, W)$. We call its degree 0 part a *truncated Jacobian algebra* and denote it by $\mathcal{P}(Q, W)_C$.

Our interest in truncated Jacobian algebras lies in the following result.

Theorem 7.2 *([8, Theorem 3.11]). If (Q, W) is a self-injective QP and C is a cut, then $\mathcal{P}(Q, W)_C$ is 2-representation finite. Moreover, every basic 2-representation finite algebra is obtained in this way.*

Now assume that a finite cyclic group G acts on $\mathcal{P}(Q, W)$ satisfying the assumptions (A1)–(A7). We want to understand when a cut in (Q_G, W_G) can be induced from one in (Q, W). We call a cut in (Q, W) invariant under the $*$ action of G a *G-invariant cut.*

Proposition 7.3. *Let C be a G-invariant cut in (Q, W). Then the subset $C_G = C_1 \cup C_2 \cup C_3 \cup C_4$ of $(Q_G)_1$ defined by*

$$C_1 = \{\tilde{\alpha} \,|\, \alpha \in C \text{ of type (1)}\}, \quad C_x = \{\tilde{\alpha}^\mu \,|\, \alpha \in C \text{ of type } (x),\ 0 \leq \mu \leq n-1\},\ x = 2, 3, 4,$$

is a cut in (Q_G, W_G).

Proof. In order to show that C_G is a cut in (Q_G, W_G), we shall prove that every cycle in W_G has degree 1 with respect to d_{C_G}. Thus we have four different cases to consider.

(i) Let $c \in \mathcal{C}(\mathrm{i})$, so $c = \alpha_1 g^{t_1}(\alpha_2) \cdots g^{t_1+\cdots+t_{l-1}}(\alpha_l)$ for some arrows $\alpha_i \in Q_1$ of type (1). Then W_G contains the cycle $\tilde{c} = \tilde{\alpha}_1 \cdots \tilde{\alpha}_l$ and, since C is G-invariant, we have

$$d_{C_G}(\tilde{c}) = \sum_{i=1}^{l} d_{C_G}(\tilde{\alpha}_i) = \sum_{i=1}^{l} d_C(\alpha_i) = \sum_{i=1}^{l} d_C(g^{t_1+\cdots+t_{i-1}}(\alpha_i)) = d_C(c) = 1.$$

(ii) Let $c \in \mathcal{C}(\mathrm{ii})$, so $c = \alpha_1\alpha_2 g^{t_2}(\alpha_3) \cdots g^{t_2+\cdots+t_{l-1}}(\alpha_l)$ for α_1 of type (3), α_2 of type (2) and $\alpha_3, \ldots, \alpha_l$ of type (1). For each $\mu = 0, \ldots, n-1$ we have a cycle $\tilde{c}^\mu = \tilde{\alpha}_1^\mu \widetilde{g^{-t_2}(\alpha_2)}^\mu \tilde{\alpha}_3 \cdots \tilde{\alpha}_l$ in W_G and

$$\begin{aligned} d_{C_G}(\tilde{c}^\mu) &= d_{C_G}(\tilde{\alpha}_1^\mu) + d_{C_G}(\widetilde{g^{-t_2}(\alpha_2)}^\mu) + \sum_{i=3}^{l} d_{C_G}(\tilde{\alpha}_i) \\ &= \sum_{i=1}^{l} d_C(\alpha_i) = \sum_{i=1}^{l} d_C(g^{t_1+\cdots+t_{i-1}}(\alpha_i)) = d_C(c) = 1. \end{aligned}$$

(iii) Let $c \in \mathcal{C}(\mathrm{iii})$, so $c = \alpha_1\alpha_2 \cdots \alpha_l$ for α_1 of type (2), α_2 of type (3) and $\alpha_3, \ldots, \alpha_l$ of type (4). For each $\mu = 0, \ldots, n-1$ we have a cycle $\tilde{c}^\mu = \tilde{\alpha}_1^\mu \tilde{\alpha}_2^{\mu-b_3} \cdots \tilde{\alpha}_{l-1}^{\mu-b_l} \tilde{\alpha}_l^\mu$ in W_G, where $b_i = b(\alpha_i) + \cdots + b(\alpha_l)$. Hence

$$d_{C_G}(\tilde{c}^\mu) = d_{C_G}(\tilde{\alpha}_1^\mu) + \sum_{i=2}^{l} d_{C_G}(\tilde{\alpha}_i^{\mu-b_{i+1}}) = \sum_{i=1}^{l} d_C(\alpha_i) = d_C(c) = 1.$$

(iv) Let $c \in \mathcal{C}(\mathrm{iv})$, so $c = \alpha_1\alpha_2 \cdots \alpha_l$ for α_i of type (4). For each $\mu = 0, \ldots, n-1$ we have a cycle $\tilde{c}^\mu = \tilde{\alpha}_1^{\mu-b_2} \tilde{\alpha}_2^{\mu-b_3} \cdots \tilde{\alpha}_{l-1}^{\mu-b_l} \tilde{\alpha}_l^\mu$ in W_G, where $b_i = b(\alpha_i) + \cdots + b(\alpha_l)$. Hence

$$d_{C_G}(\tilde{c}^\mu) = \sum_{i=1}^{l} d_{C_G}(\tilde{\alpha}_i^{\mu-b_{i+1}}) = \sum_{i=1}^{l} d_C(\alpha_i) = d_C(c) = 1. \qquad \square$$

Observe that from [14, Corollary 1.6(1)], 2-representation finiteness is preserved by taking skew group algebras. Thus it follows from Theorem 7.2 that the property of being a truncated Jacobian algebra is also preserved. In our setting, the corresponding cut on (Q_G, W_G) is precisely C_G:

Proposition 7.4. *Let C be a G-invariant cut in (Q, W) and let C_G be the cut constructed in Proposition 7.3. Then the action of G on $\mathcal{P}(Q, W)$ restricts to an action on $\mathcal{P}(Q, W)_C$, and the skew group algebra $(\mathcal{P}(Q, W)_C)G$ is Morita equivalent to $\mathcal{P}(Q_G, W_G)_{C_G}$.*

Proof. Call $\Lambda = \mathcal{P}(Q, W)$ and let Λ_0 be its degree 0 part with respect to the grading d_C, so $\Lambda_0 = \mathcal{P}(Q, W)_C$. The fact that C is G-invariant implies that G preserves the grading, so the first assertion holds.

Now note that we can define a grading on ΛG by assigning degree $d_C(x)$ to $x \otimes h$ for all $h \in G$ and all homogeneous elements $x \in \Lambda$. Moreover this induces a grading on $\eta(\Lambda G)\eta$ and we have that $(\eta(\Lambda G)\eta)_0 = \eta(\Lambda_0 G)\eta$. Hence, in order to prove the claim, it is enough to show that the grading on $\eta(\Lambda G)\eta$ coincides with the grading d_{C_G} on $\mathcal{P}(Q_G, W_G)$ under the isomorphism $\eta(\Lambda G)\eta \cong \mathcal{P}(Q_G, W_G)$. But this follows immediately from the definition of C_G, since both algebras are generated in degree 0 and 1 and the elements of degree 1 in $\eta(\Lambda G)\eta$ are exactly the ones given by C_G. $\square$

Let (Q, W) be a self-injective QP with a group G acting as per the assumptions (A1)–(A7). Then (Q_G, W_G) is self-injective, so its truncated Jacobian algebras are 2-representation finite. In the spirit of [8, §7], we will give sufficient conditions on (Q, W) for the truncated Jacobian algebras of (Q_G, W_G) to be derived equivalent to each other.

In the following discussion we do not need to assume self-injectivity.

Definition 7.5. We say that (Q, W) *has enough cuts* if every arrow of Q is contained in a cut. We say that (Q, W) *has enough G-invariant cuts* if every arrow of Q is contained in a G-invariant cut (cf. [8, Definition 7.4]).

Lemma 7.6. *If (Q, W) has enough G-invariant cuts, then (Q_G, W_G) has enough cuts.*

Proof. Let $\beta \in (Q_G)_1$, so $\beta = \tilde{\alpha}$ or $\beta = \tilde{\alpha}^\mu$ for some $\alpha \in Q_1$. Let C be a G-invariant cut in (Q, W) containing α, then the cut C_G in (Q_G, W_G) constructed in Proposition 7.3 contains β. $\square$

To use the results of [8], we need to study the topology of the canvas of (Q_G, W_G). We will do this in the case of G acting by rotations on a strongly planar QP.

Proposition 7.7. *Let (Q, W) be a strongly planar QP with a group G acting by rotations, and assume that there is a vertex of Q fixed by G. Then $X_{(Q_G, W_G)}$ is simply connected.*

Proof. Let us decompose $X_{(Q,W)} = \mathcal{U} \cup \mathcal{V}$, where $\mathcal{V}$ is the subcomplex consisting of all the faces adjacent to the central vertex, and $\mathcal{U}$ is the subcomplex consisting of the other faces. Since (Q, W) is strongly planar, $X_{(Q,W)}$ is homeomorphic to a disk. Note that if

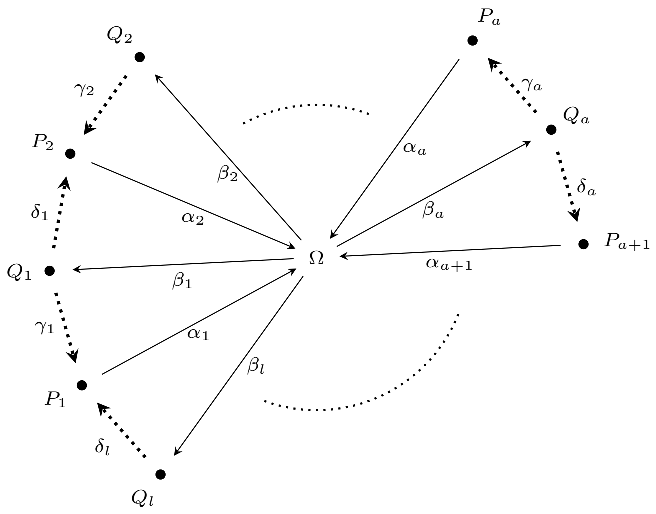

Fig. 1. The subcomplex $\mathcal{V}$ of $X_{(Q,W)}$.

G is trivial, then the statement is immediate. Otherwise, this implies that the central vertex Ω has a neighbourhood in $X_{(Q,W)}$ which is itself homeomorphic to a disk. So $\mathcal{V}$ is homeomorphic to a disk as well. Thus $\mathcal{V}$ looks as in Fig. 1, where α_i, β_i are arrows, γ_i, δ_i are paths, and all cycles $\alpha_i\gamma_i\beta_i$, $\alpha_{i+1}\delta_i\beta_i$, and $\alpha_1\delta_l\beta_l$ bound faces. The action of a generator g of G is given by adding a to indices. By picking g suitably, we can assume that $an = l$, where $n = |G|$. We choose as representatives of vertices a set $\mathcal{E}$ which contains $\{\Omega, P_1, \dots, P_a, Q_1, \dots, Q_a\}$. Observe that G acts freely on $\mathcal{U}$, and it also acts freely on $\mathcal{U} \cap \mathcal{V}$. Then $X_{(Q_G,W_G)} = \tilde{\mathcal{U}} \cup \tilde{\mathcal{V}}$, where $\tilde{\mathcal{V}}$ is as in Fig. 2 and $\tilde{\mathcal{U}} \cong \mathcal{U}/G$ is the quotient space of $\mathcal{U}$ by G, by our construction of Q_G. In the picture we denote by $\tilde{\delta}_i$ the product of $\tilde{d}$, where d is an arrow of δ_i, and similarly for $\tilde{\gamma}_i$. We have that $\tilde{\mathcal{U}}$ is attached to $\tilde{\mathcal{V}}$ along $(\mathcal{U} \cap \mathcal{V})/G = \tilde{\mathcal{U}} \cap \tilde{\mathcal{V}}$. Now observe that since $X_{(Q,W)}$ is simply connected, it must retract to $\mathcal{V}$. In particular there is a deformation retraction F between $\mathcal{U}$ and $\mathcal{U} \cap \mathcal{V}$. We choose F such that it commutes with the action of G on $\mathcal{U}$. Then there is an induced deformation retraction $\tilde{F}$ between $\tilde{\mathcal{U}}$ and $\tilde{\mathcal{U}} \cap \tilde{\mathcal{V}}$. In particular $X_{(Q_G,W_G)}$ retracts to $\tilde{\mathcal{V}}$, so they have the same homotopy type.

We need to describe the faces of $\tilde{\mathcal{V}}$. Let us look at the set of cycles in W involving only vertices in $\mathcal{V}$. These are $\gamma_1\beta_1\alpha_1, \dots, \gamma_a\beta_a\alpha_a, \delta_1\beta_1\alpha_2, \dots, \delta_a\beta_a\alpha_{a+1}$ and their orbits. These cycles are all of type (ii), so we have

$$\begin{aligned} \widetilde{\gamma_i\beta_i\alpha_i}^{\,\mu} &= \tilde{\gamma}_i\tilde{\beta}_i^{\mu}\tilde{\alpha}_i^{\mu} \\ \widetilde{\delta_i\beta_i\alpha_{i+1}}^{\,\mu} &= \tilde{\delta}_i\tilde{\beta}_i^{\mu}\tilde{\alpha}_{i+1}^{\mu}, \end{aligned}$$

for $i = 1, \dots, a$, with the notation $\tilde{\alpha}_{a+1}^{\mu} = \tilde{\alpha}_1^{\mu}$. Now fix $\mu \in \{0, \dots, n-1\}$. Then Ω^{μ} is contained in every $\tilde{\gamma}_i\tilde{\beta}_i^{\mu}\tilde{\alpha}_i^{\mu}$, in every $\tilde{\delta}_i\tilde{\beta}_i^{\mu}\tilde{\alpha}_{i+1}^{\mu}$, and no other cycle in W_G. The subcomplex consisting of the faces corresponding to these $2a$ cycles is a disk with centre Ω^{μ}. Thus

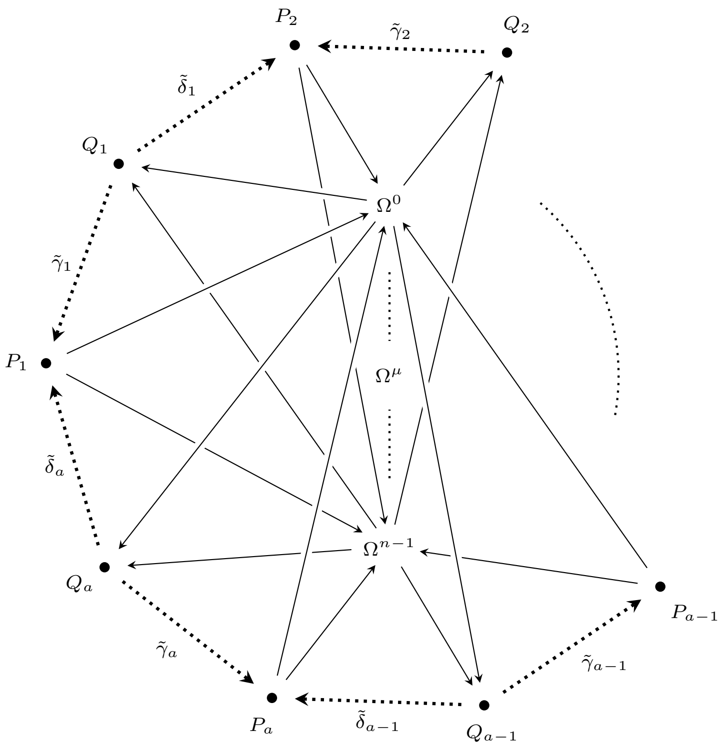

Fig. 2. The subcomplex $\tilde{\mathcal{V}}$ of $X_{(Q_G,W_G)}$.

$\tilde{\mathcal{V}}$ consists of n disks glued along their boundary $\tilde{\delta}_a \cdots \tilde{\gamma}_2^{-1}\tilde{\delta}_1\tilde{\gamma}_1^{-1}$, and therefore has the homotopy type of a bouquet of spheres. In particular it is simply connected, which concludes the proof. $\square$

In Example 8.3 we proceed as in the proof of Proposition 7.7 to determine the canvas of (Q_G, W_G).

Remark 7.8. If G acts on a planar QP (Q, W) by rotations and (Q, W) has a G-invariant cut, then Q must have a central vertex. Indeed, Q has either a central vertex or a central cycle, but on a central cycle one cannot choose exactly one arrow in a way which is invariant under rotations.

In the self-injective case we have the following result.

Theorem 7.9. *Let (Q, W) be a strongly planar self-injective QP, with a group G acting by rotations and enough G-invariant cuts. Then all the truncated Jacobian algebras of (Q_G, W_G) are derived equivalent to each other.*

Proof. By Lemma 7.6, (Q_G, W_G) has enough cuts. By Proposition 7.7, $X_{(Q_G,W_G)}$ is simply connected. Then we conclude by [8, Theorem 8.7]. $\square$

In particular, note that this result applies to QPs coming from Postnikov diagrams, provided they have enough G-invariant cuts. It should be noted that we know of no examples of a self-injective QP with a cut that does not have enough cuts, nor of a self-injective QP with a G-invariant cut that does not have enough G-invariant cuts.

8. Examples

In this section we will illustrate our construction with some examples. For simplicity we will assume that $k = \mathbb{C}$, so the assumption (A1) will be always satisfied.

8.1. Examples from planar rotation-invariant QPs

As we have seen in Section 6, many examples where our construction may be applied are given by quivers embedded in the plane with a group acting by rotations. Let us illustrate some of them.

Example 8.1 *(2-Representation finite algebras of type A).* A family of examples of self-injective planar QPs is given by 3-preprojective algebras of 2-representation finite algebras of type A, which were introduced in [11] and are defined as follows.

Let $s \geq 1$ and $Q = Q^{(s)}$ be the quiver defined by

$$Q_0 = \{(x_1, x_2, x_3) \in \mathbb{Z}^3_{\geq 0} \mid x_1 + x_2 + x_3 = s - 1\},$$
$$Q_1 = \{\alpha_i \colon x \to x + f_i \mid 1 \leq i \leq 3,\, x, x + f_i \in Q_0\},$$

where $f_1 = (-1, 1, 0)$, $f_2 = (0, -1, 1)$, $f_3 = (1, 0, -1)$. The potential W is given by the sum of all cycles of the form $\alpha_1\alpha_2\alpha_3$ minus the ones of the form $\alpha_1\alpha_3\alpha_2$.

The Nakayama automorphism of $\Lambda = \mathcal{P}(Q, W)$ is induced by the unique automorphism of Q given on vertices by $(x_1, x_2, x_3) \mapsto (x_3, x_1, x_2)$. Then the group G generated by it acts on Q by an anticlockwise rotation by $2\pi/3$. We may note that this action has a (unique) fixed vertex if and only if $s \equiv 1 \pmod 3$. In that case the vertex $(\frac{s-1}{3}, \frac{s-1}{3}, \frac{s-1}{3})$ is fixed.

Proposition 8.2. *If $s \equiv 1 \pmod 3$, then $Q^{(s)}$ has enough G-invariant cuts.*

Proof. Call $x_0 = (\frac{s-1}{3}, \frac{s-1}{3}, \frac{s-1}{3})$ the unique fixed vertex. Let $L = \{(x_1, x_2, x_3) \in \mathbb{Z}^3 \mid x_1 + x_2 + x_3 = 0\}$ and note that it is a free abelian group of rank 2 with basis $\{f_1, f_2\}$. We may embed Q_0 in L via the map $x \mapsto x - x_0$. Note that the action of G on Q_0 can be naturally extended to an action on L, which is again given by $(x_1, x_2, x_3) \mapsto (x_3, x_1, x_2)$.

Let $\omega \colon L \to \mathbb{Z}/3\mathbb{Z}$ be the group homomorphism defined by $\omega(f_i) = 1$ for $i = 1, 2, 3$. For each $j \in \mathbb{Z}/3\mathbb{Z}$ we define the following subset of Q_1:

$$C_j = \{\alpha_i \colon x \to x + f_i \mid \omega(x - x_0) = j\}.$$

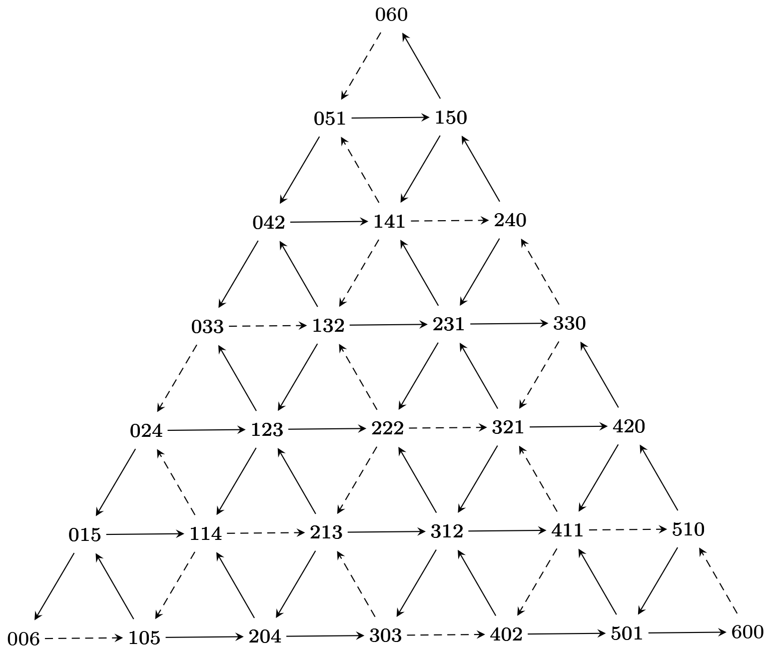

Fig. 3. The quiver $Q^{(7)}$. The cut C_0 is given by the dashed arrows.

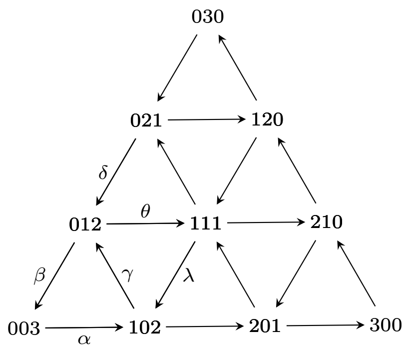

Fig. 4. The quiver $Q^{(4)}$.

Then C_j is a cut (cf. [9, Example 5.8]). It is symmetric because ω is invariant on G-orbits. Moreover every arrow is contained in a cut of this type, so the statement follows. $\square$

As an example, we illustrate the cut C_0 of $Q^{(7)}$ in Fig. 3.

Now we will describe our skew group algebra construction for the quiver $Q = Q^{(4)}$ (which is depicted in Fig. 4).

We can choose, for example, $\mathcal{E} = \{(0,0,3),(0,1,2),(1,0,2),(1,1,1)\}$ as a set of representatives of vertices. For simplicity we shall denote the elements of this set by $\{1,2,3,4\}$ respectively. Then Q_G (depicted in Fig. 5) has vertices $\eta^1, \eta^2, \eta^3, \eta^4_0, \eta^4_1, \eta^4_2$, which will be

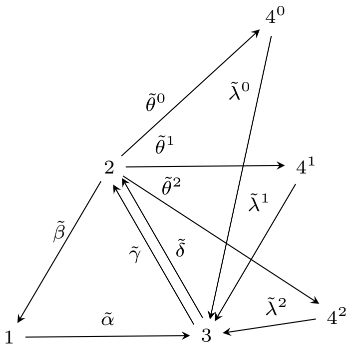

Fig. 5. The quiver $Q_G^{(4)}$.

denoted respectively by $1, 2, 3, 4^0, 4^1, 4^2$. We will also rename the arrows of type (1), (2), (3) in Q. These are

$$\alpha\colon 1 \to 3, \quad \beta\colon 2 \to 1, \quad \gamma\colon 3 \to 2, \quad \delta\colon g^2(3) \to 2 \quad \text{of type (1)},$$
$$\theta\colon 2 \to 4 \quad \text{of type (2)},$$
$$\lambda\colon 4 \to 3 \quad \text{of type (3)}.$$

We take $\mathcal{C} = \{c_1, c_2, c_3\}$, where $c_1 = \alpha\beta\gamma$ is of type (i) and $c_2 = \lambda\theta\gamma, c_3 = \lambda g(\theta)g(\delta)$ are of type (ii). Note that $p(c_2) = 0$ and $p(c_3) = 1$. Then we get

$$W_G = -\tilde{\alpha}\tilde{\beta}\tilde{\gamma} + \sum_{\mu=0}^{2} \tilde{\lambda}^\mu \tilde{\theta}^\mu \tilde{\gamma} - \sum_{\mu=0}^{2} \zeta^{-\mu} \tilde{\lambda}^\mu \tilde{\theta}^\mu \tilde{\delta}.$$

By the results in Section 5, the dual group $\hat{G} = \langle \chi \rangle$ acts on Q_G as follows. The vertices $1, 2, 3$ are fixed, while $\chi(4^\mu) = 4^{\mu+1}$, $\mu = 0, 1, 2$. The arrows $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ are fixed, $\chi(\tilde{\theta}^\mu) = \tilde{\theta}^{\mu+1}$ and $\chi(\tilde{\lambda}^\mu) = \tilde{\lambda}^{\mu+1}$, $\mu = 0, 1, 2$. Since $t(\delta) = 2$, we have $\chi(\tilde{\delta}) = \zeta^2\tilde{\delta}$. Note that, in the process of getting back the initial quiver using the isomorphism ϕ of Proposition 5.3, the vertices $4^0, 4^1, 4^2$ give rise to the vertex $(1, 1, 1)$ of Q, the vertex 2 gives rise to $(0, 1, 2), (1, 2, 0), (2, 0, 1)$, 3 to $(0, 2, 1), (2, 1, 0), (1, 0, 2)$ and 1 to $(0, 0, 3), (0, 3, 0), (3, 0, 0)$.

Example 8.3 *(Self-injective QPs from Postnikov diagrams).* In this example we illustrate Corollary 6.10 and (the proof of) Proposition 7.7. Let Q be the quiver of Fig. 6, with the potential W given by the sum of the clockwise faces minus the sum of the anticlockwise faces. Thus (Q, W) is a strongly planar quiver with potential. It is constructed from a rotation-invariant $(4, 16)$-Postnikov diagram, see [16, Figure 19]. By Theorem 6.6, its Jacobian algebra Λ is therefore self-injective, with Nakayama automorphism φ induced by a rotation by $\frac{\pi}{2}$. Let us consider the group $G = \langle \varphi^2 \rangle$. Then the skew group algebra ΛG is Morita equivalent to the Jacobian algebra $\mathcal{P}(Q_G, W_G)$, where Q_G is depicted in Fig. 7. The canvas $X_{(Q_G, W_G)}$ is given by an octahedron in the middle attached to an annulus

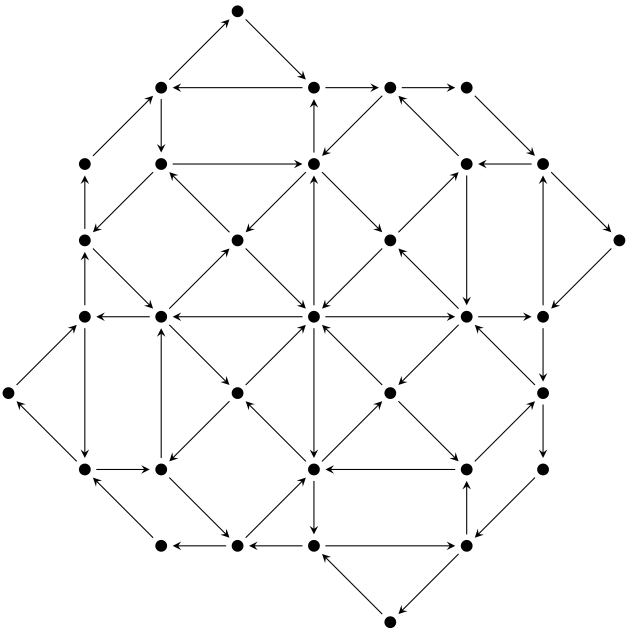

Fig. 6. A self-injective QP with Nakayama automorphism φ of order 4.

made of all the remaining faces. Note that this describes the potential W_G completely up to signs. This algebra is self-injective with Nakayama automorphism given by $\varphi \otimes 1$, but it is not symmetric since its Nakayama permutation has order 2.

If we instead take the skew group algebra construction with respect to $\langle\varphi\rangle$, we get the quiver of Fig. 8. Its canvas is an annulus consisting of the outer cycles, attached to four disks sharing their boundary circle. These disks are subdivided into two triangles each. Again note that describing the canvas determines the potential up to fourth roots of unity. This algebra is symmetric by Corollary 6.10.

8.2. Examples from tensor products of quivers

The following family of self-injective QPs was introduced in [8, §5.2]. Let us recall their definition.

Given two quivers Q^1, Q^2 without oriented cycles we can define a new quiver $Q = Q^1 \tilde{\otimes} Q^2$ with $Q_0 = Q_0^1 \times Q_0^2$ and $Q_1 = (Q_0^1 \times Q_1^2) \sqcup (Q_1^1 \times Q_1^2) \sqcup (Q_1^1 \times Q_0^2)$. The starting and ending points of the arrows of Q are given by

$$\mathfrak{s}(\alpha, y) = (\mathfrak{s}(\alpha), y), \quad \mathfrak{s}(x, \beta) = (x, \mathfrak{s}(\beta)), \quad \mathfrak{s}(\alpha, \beta) = (\mathfrak{t}(\alpha), \mathfrak{t}(\beta)),$$

$$\mathfrak{t}(\alpha, y) = (\mathfrak{t}(\alpha), y), \quad \mathfrak{t}(x, \beta) = (x, \mathfrak{t}(\beta)), \quad \mathfrak{t}(\alpha, \beta) = (\mathfrak{s}(\alpha), \mathfrak{s}(\beta)),$$

for $x \in Q_0^1$, $y \in Q_0^2$, $\alpha \in Q_1^1$, $\beta \in Q_1^2$. We define a potential on Q by

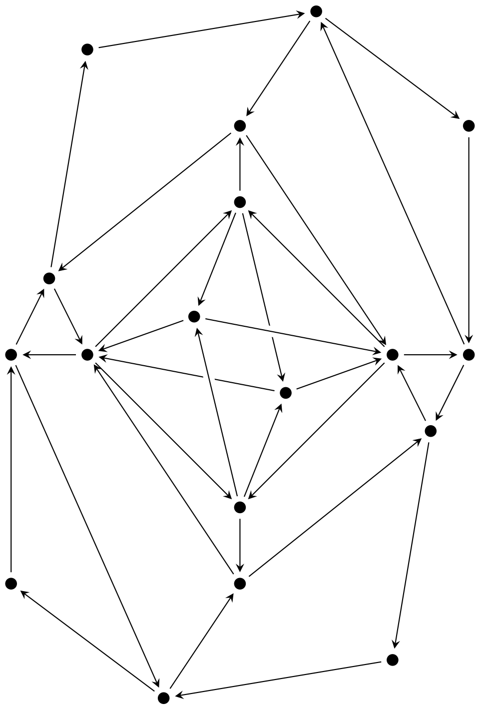

Fig. 7. The quiver of the skew group algebra $\Lambda\langle\varphi^2\rangle$.

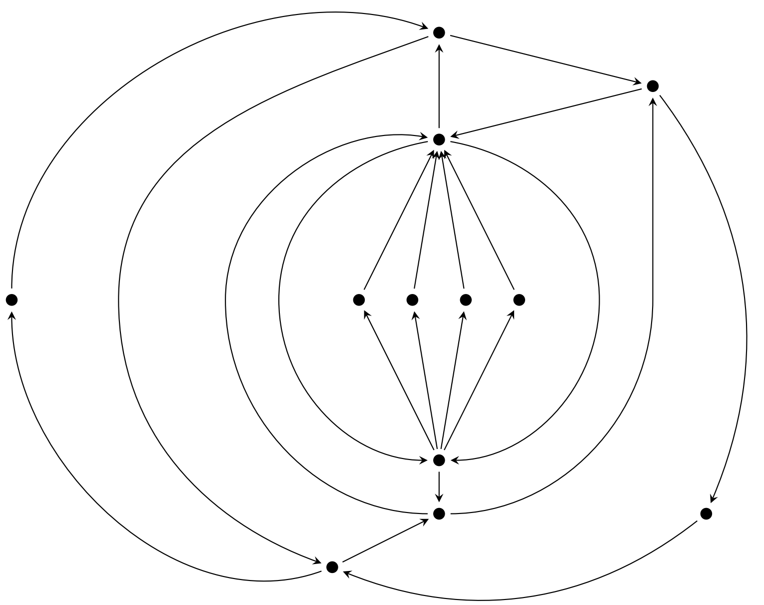

Fig. 8. The quiver of the skew group algebra $\Lambda\langle\varphi\rangle$.

$$W = W^{\tilde{\otimes}}_{Q^1,Q^2} = \sum_{\alpha \in Q^1_1, \beta \in Q^2_1} (\alpha, \mathfrak{t}(\beta))(\mathfrak{s}(\alpha), \beta)(\alpha, \beta) - (\mathfrak{t}(\alpha), \beta)(\alpha, \mathfrak{s}(\beta))(\alpha, \beta).$$

Now we consider group actions on kQ. Let $G_1 = \langle g_1 \rangle$ and $G_2 = \langle g_2 \rangle$ be finite cyclic groups and suppose that the following condition holds:

$(*)$ either one of G_1 or G_2 is trivial, or $G_1 \cong G_2$.

We denote by n the maximum of the orders of G_1 and G_2. Let G be the subgroup of $G_1 \times G_2$ generated by (g_1, g_2), and note that it is cyclic of order n.

Lemma 8.4. *Let Q^1, Q^2, G_1, G_2 as above. Suppose we have actions of G_i on kQ^i, $i = 1, 2$, which satisfy the assumptions (A1), (A2), (A3), (A6), and:*

(A3') *every arrow in Q^i between two fixed vertices is fixed by G_i.*

Then the induced action of G on kQ satisfies the assumptions (A1)–(A7).

Proof. Assumption (A1) holds by the assumptions on the orders of G_1 and G_2. The assumptions (A2), (A5) and (A3) follow immediately by hypothesis. Now note that G permutes the cycles of the potential $W^{\tilde{\otimes}}_{Q^1,Q^2}$, and every cycle is sent to a cycle with the same coefficient. Hence $GW = W$. Finally, assumption (A7) is satisfied because all cycles have length 3. □

If Q^1 and Q^2 are Dynkin quivers with the same Coxeter number and which are stable under their canonical involutions (see [8, §5.2] for definitions), then $(Q, W) = (Q^1 \tilde{\otimes} Q^2, W^{\tilde{\otimes}}_{Q^1,Q^2})$ is a self-injective QP by [8, Proposition 5.1]. Let g_1 and g_2 be the unique automorphisms of, respectively, Q^1 and Q^2 given by extending to arrows their canonical involutions.

Proposition 8.5. *Let Q^1 and Q^2 be Dynkin quivers which are stable under their canonical involutions and have the same Coxeter number. Let G be the cyclic group generated by (g_1, g_2) and consider the induced action of G on $Q = Q^1 \tilde{\otimes} Q^2$. Then (Q_G, W_G) is a self-injective QP with enough cuts.*

Proof. Note that g_1 and g_2 have order either 1 or 2, so the condition $(*)$ for $G_1 = \langle g_1 \rangle$ and $G_2 = \langle g_2 \rangle$ is satisfied. The assumptions (A1), (A2), (A3'), (A3), and (A6) for G_1 and G_2 are immediately checked, so by Lemma 8.4 we can apply the construction of Section 3 to (Q, W) and G. By [8, Proposition 5.1] (Q, W) is self-injective, hence so is (Q_G, W_G).

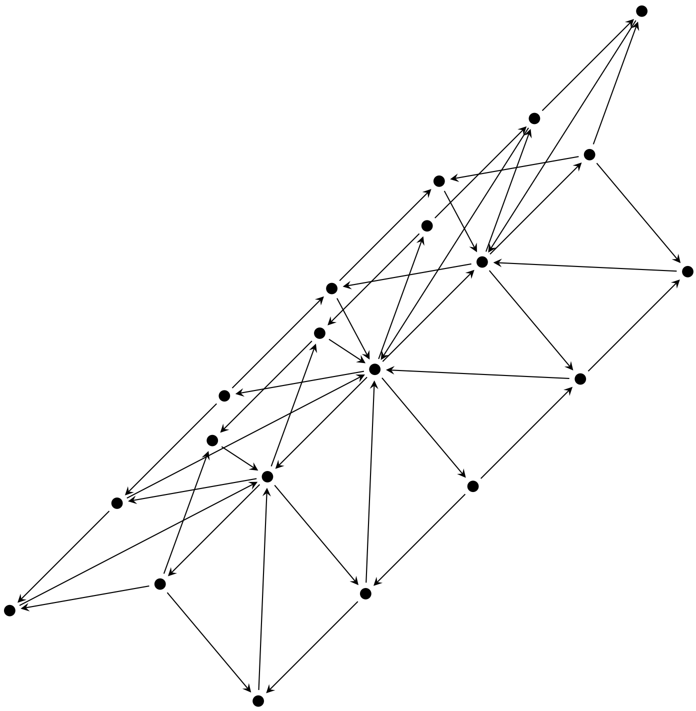

Fig. 9. The quiver $Q^1 \tilde{\otimes} Q^2$.

From the definition of $W^{\tilde{\otimes}}_{Q^1,Q^2}$ it follows that the subsets (Q^1_0, Q^2_1), (Q^1_1, Q^2_1) and (Q^1_1, Q^2_0) of Q_1 are all G-invariant cuts. Since every arrow of Q is contained in one of them, we have that (Q, W) has enough G-invariant cuts. Hence (Q_G, W_G) has enough cuts by Lemma 7.6. $\square$

Example 8.6. Consider the following Dynkin quivers:

$$Q^1: \quad \bullet \longleftarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \qquad\qquad Q^2: \quad \begin{tikzcd} & & \bullet \\ \bullet & \bullet \arrow[l] \arrow[ur] \arrow[dr] & \\ & & \bullet \end{tikzcd}$$

Here Q^1 is of type A_5 and Q^2 of type D_4, so they have the same Coxeter number. The canonical involution of Q^1 is the reflection with respect to the central vertex, while the one of Q^2 is the identity. Hence the two quivers are stable and, by Proposition 8.5, (Q_G, W_G) is a self-injective QP with enough cuts. The quivers Q and Q_G are illustrated respectively in Figs. 9 and 10.

All examples we have illustrated so far are related to self-injective QPs. In the next one we will consider a case where the QP we start with is not self-injective.

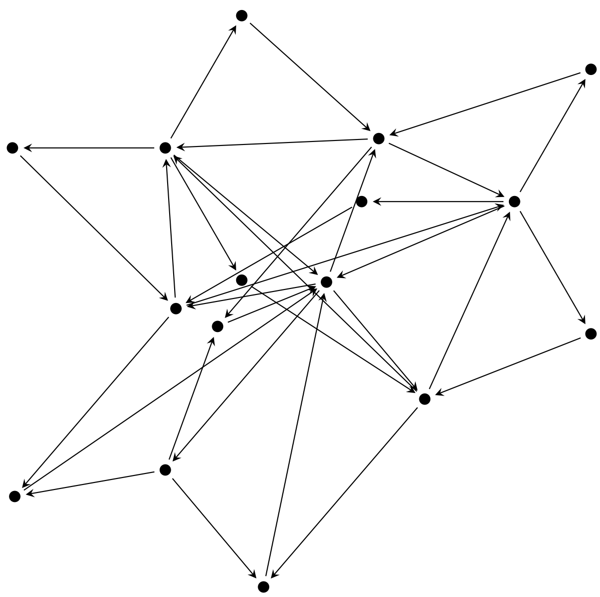

Fig. 10. The quiver $(Q^1\tilde{\otimes}Q^2)_G$.

Example 8.7. Consider the Dynkin quivers

$$Q^1: \quad 2 \xleftarrow{\beta} 1 \xleftarrow{\alpha} 0 \xrightarrow{\alpha'} 1' \xrightarrow{\beta'} 2' \qquad\qquad Q^2: \quad 0 \xleftarrow{\gamma_1} 1 \xleftarrow{\gamma_2} 2$$

and let $Q = Q^1\tilde{\otimes}Q^2$ (see Fig. 11).

Let g be the unique automorphism of Q^1 given on vertices by $g(0) = 0$, $g(i) = i'$ and $g(i') = i$, $i = 1, 2$. Then we can consider the action of the cyclic group $G = \langle(g, \mathrm{id})\rangle$ of order 2 on Q. If we apply the construction of Section 3 choosing as a set of representatives of the vertices $\mathcal{E} = \{(i, j) \,|\, i, j = 0, 1, 2\}$, then we obtain the quiver Q_G of Fig. 12. We can take

$$\mathcal{C} = \{(\alpha, i-1)(0,\gamma_i)(\alpha,\gamma_i), (1,\gamma_i)(\alpha, i)(\alpha,\gamma_i), (\beta, i-1)(0,\gamma_i)(\beta,\gamma_i), (1,\gamma_i)(\beta, i)(\beta,\gamma_i) \,|\, i = 1, 2\}$$

and obtain the potential

$$W_G = \sum_{i=1}^{2} \widetilde{(\beta, i-1)}\widetilde{(0,\gamma_i)}\widetilde{(\beta,\gamma_i)} - \widetilde{(1,\gamma_i)}\widetilde{(\beta, i)}\widetilde{(\beta,\gamma_i)} +$$
$$+ \sum_{i=1}^{2}\sum_{\mu=0}^{1} \widetilde{(\beta, i-1)}\widetilde{(0,\gamma_i)}^{\mu}\widetilde{(\beta,\gamma_i)} - \widetilde{(1,\gamma_i)}^{\mu}\widetilde{(\beta, i)}\widetilde{(\beta,\gamma_i)}.$$

Remark 8.8. We may choose another basis for $\operatorname{rad}\mathcal{P}(Q,W)/\operatorname{rad}^2\mathcal{P}(Q,W)$ by replacing (α', i) with $-(\alpha', i)$ and (β', i) with $-(\beta', i)$, $i = 0, 1, 2$. In this way we get that $\mathcal{P}(Q,W) \cong$

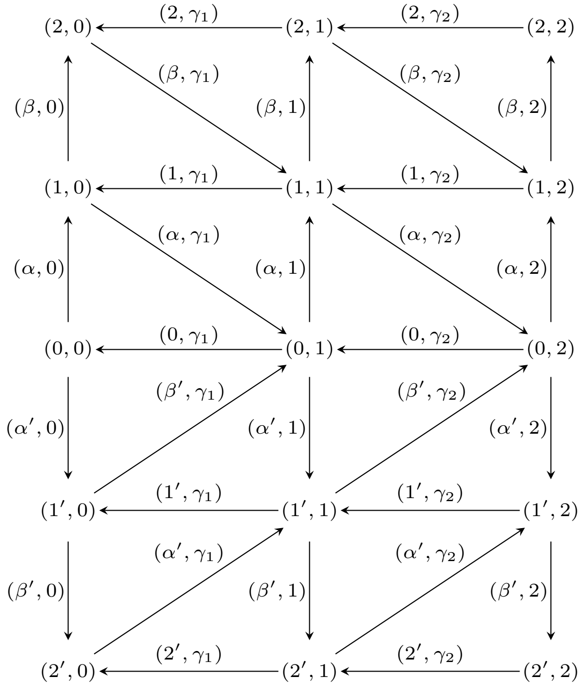

Fig. 11. The quiver $Q^1 \tilde{\otimes} Q^2$.

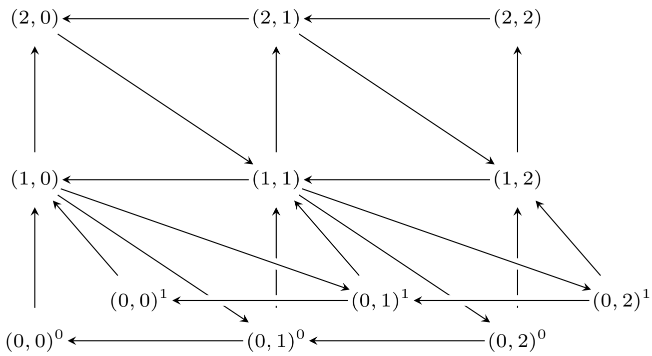

Fig. 12. The quiver $(Q^1 \tilde{\otimes} Q^2)_G$.

$\mathcal{P}(Q, W')$, where W' is the potential defined as the sum of all the clockwise 3-cycles minus the sum of all the anticlockwise ones. We have an action of G on $\mathcal{P}(Q, W')$ such that $\mathcal{P}(Q, W)G \cong \mathcal{P}(Q, W')G$, but note that in this case the assumption (A6) is no longer satisfied.

Now let us consider the G-invariant cut $C = Q^0 \times Q^1$ in Q. We may note that the truncated Jacobian algebra $\mathcal{P}(Q, W')_C$ is isomorphic to the Auslander algebra of Q^1.

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